ALGEBRAIC STRUCTURES

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Examination 30th August 2013

Solutions. Complete solutions are required for each problem.

Marking. Each problem is worth 6 points.

- The marks 3, 4, and 5 correspond approximately to the scores 18, 25, and 32, respectively, distributed *reasonably* evenly among the three divisions Group Theory, Ring Theory, and Field Theory.
- Also, in order to pass, a student should demonstrate some knowledge of the fundamental definitions of the course. Definitions should be written out formally, using complete sentences.

1. Consider the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 4 & 3 & 1 & 5 \end{pmatrix}.$$

   (a) Write π in cycle notation.

   (b) Determine whether π is even or odd.

   (c) Find a permutation commuting with π, which is neither π itself nor the identity permutation.

   (d) Find a permutation *not* commuting with π.

   (e) Determine the order of the subgroup generated by π.

2. (a) Starting from the concept of an integral domain and an irreducible element, define a *unique factorisation domain*.

   (b) Consider the polynomial

$$p(x) = x^4 + x^3 + 5x^2 + 10x + 5.$$

   Factorise $p(x)$ into irreducible factors over $\mathbf{Q}$.

(c) Factorise $p(x)$ into irreducibles over $\mathbf{Z}_3$.

3. (a) Define the notion of a *group action*.

   (b) Consider the set of four matrices:

$$G = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}.$$

   Shew that G is a group under matrix multiplication. Which well-known group is it isomorphic to?

   (c) Shew that G acts on the plane $\mathbf{R}^2$ by left multiplication: $A \cdot x = Ax$, for $A \in G$ and $x \in \mathbf{R}^2$.

   (d) What is the orbit of a point $P = (p, q)$ under this action? What is the stabiliser?

4. (a) Define the concept of a *ring*.

   (b) Define *ring homomorphisms* and *ring isomorphisms*.

   (c) Shew that the rings $\mathbf{Q}[x]/(x^2 - 1)$ and $\mathbf{Q} \times \mathbf{Q}$ are isomorphic.

5. (a) Define a *soluble group*.

   (b) Is S_3 soluble?

   (c) Is $\mathbf{Z}$ soluble?

6. (a) Define what it means for P to be a *prime ideal* of a commutative, unital ring R. What is known about the structure of the factor ring R/P when P is a prime ideal?

   (b) Let R be a commutative, unital ring, let S be a subring, and let I be an ideal of R. Prove that, if $S \cap I = \{0\}$, then the set

$$T = \{\, s + I \mid s \in S \,\}$$

   forms a subring of R/I isomorphic to S.

7. (a) Define the *Galois group* of a polynomial over a field F.

   (b) Factorise $p(x) = x^4 - 4$ into irreducibles over $\mathbf{Q}$.

   (c) Determine the Galois group of $p(x)$ over $\mathbf{Q}$.