ALGEBRAIC STRUCTURES

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1. (a) $\pi = (165)(34)$.
   (b) $\pi = (15)(16)(34)$ is odd.
   (c) For instance $\pi^2 = (156)$.
   (d) For instance $\rho = (16)$, since
   $$\pi\rho = (15)(34) \neq (56)(34) = \rho\pi.$$
   (e) The order of π is $\text{LCM}(2, 3) = 6$.
2. (a) —
   (b) The factorisation is $p(x) = (x + 1)q(x)$, where
   $$q(x) = x^3 + 5x + 5$$
   is irreducible over $\mathbf{Q}$ by Eisenstein's Criterion.
   (c) The factorisation $p(x) = (x + 1)q(x)$ is still valid, but now the irreducibility of $q(x)$ follows from $q(0) = q(\pm 1) = -1$.
3. (a) —
   (b) Clearly G is closed under matrix multiplication. It contains the identity matrix I, and each matrix is its own inverse, since
   $$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
   This property shews that G is isomorphic with the Klein four-group.

(c) This is clear, since

$$A \cdot (B \cdot x) = A(Bx) = (AB)x = AB \cdot x$$

and

$$I \cdot x = Ix = x.$$

(d) The orbit of $P = (p, q)$ is $\{\, (\pm p, \pm q) \,\}$. The stabiliser is as follows.

If $p \neq 0 \neq q$: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

If $p = 0 \neq q$: $\left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

If $p \neq 0 = q$: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$

If $p = 0 = q$: $\left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} = G$

4. (a) —

(b) —

(c) Define an isomorphism by

$$\begin{aligned} \varphi \colon \mathbf{Q} \times \mathbf{Q} &\to \mathbf{Q}[x]/(x^2 - 1) \\ (a, b) &\mapsto a\frac{1 + x}{2} + b\frac{1 - x}{2} = \frac{a + b}{2} + \frac{a - b}{2}x. \end{aligned}$$

This map is clearly additive, and $\varphi(1, 1) = 1$. Multiplicativity follows since

$$\begin{aligned} \varphi(a, b)\varphi(c, d) &= \left(\frac{a + b}{2} + \frac{a - b}{2}x\right)\left(\frac{a + b}{2} + \frac{a - b}{2}x\right) \\ &= \frac{ac + bd}{2} + \frac{ac - bd}{2}x = \varphi(ac, bd). \end{aligned}$$

Since $\varphi(p + q, p - q) = p + qx$, the map φ is surjective. It is also injective since $0 = \frac{a+b}{2} + \frac{a-b}{2}x$ implies $a + b = a - b = 0$, and hence $a = b = 0$.

5. (a) —

(b) Yes: S_3 has a subnormal series

$$\{()\} \trianglelefteq \{(), (123), (321)\} \trianglelefteq S_3.$$

The factor groups are abelian of orders 3 and 2, respectively.

(c) Yes: $\mathbf{Z}$ has the trivial subnormal series

$$\{0\} \trianglelefteq \mathbf{Z}$$

with the single factor group abelian.

6. (a) —

   (b) Define an isomorphism

$$\psi\colon S \to T, \qquad s \mapsto s + I.$$

It is additive and multiplicative, and sends 1 to $1 + I$. Moreover, it is clearly surjective.

It only remains to shew that ψ is injective. Suppose $s \in S$ is such that $0 + I = \psi(s) = s + I$. This implies $s \in I$, and therefore $s \in S \cap I = \{0\}$.

7. (a) —

   (b) $p(x) = x^4 - 4 = (x^2 + 2)(x^2 - 2)$.

   (c) The splitting field of $p(x)$ is $\mathbf{Q}(\sqrt{2}, \sqrt{-2}) = \mathbf{Q}(\sqrt{2}, i)$. Any automorphism of this field must map $\sqrt{2} \mapsto \pm\sqrt{2}$ and $i \mapsto \pm i$, which leaves only four choices. On the other hand, the Conjugation Isomorphism Theorem ensures the existence of automorphisms performing all these exchanges.

(The automorphisms can of course also be constructed directly: There is

$$\zeta\colon \mathbf{Q}(i)(\sqrt{2}) \to \mathbf{Q}(i)(\sqrt{2}), \quad a + b\sqrt{2} \mapsto a - b\sqrt{2}$$

swapping $\pm\sqrt{2}$ and

$$\eta\colon \mathbf{Q}(\sqrt{2})(i) \to \mathbf{Q}(\sqrt{2})(i), \quad a + bi \mapsto a - bi$$

swapping $\pm i$, and then there is the product $\zeta\eta = \eta\zeta$ swapping both.)

Consequently, the Galois group of $p(x)$ is isomorphic with the Klein four-group.