ALGEBRAIC STRUCTURES

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Examination 16th December 2013

Solutions. Complete solutions are required for each problem.

Marking. Each problem is worth 6 points.

- The marks 3, 4 and 5 correspond approximately to the scores 18, 25 and 32, respectively, distributed *reasonably* evenly among the three divisions Group Theory, Ring Theory and Field Theory.
- Also, in order to pass, a student should demonstrate some knowledge of the fundamental definitions of the course. Definitions should be written out formally, using complete sentences.

1. (a) Define the *order* of a group element.
   (b) What is the order of 48 in the additive group $\mathbf{Z}_{60}$?
   (c) Find all elements of the subgroup of $\mathbf{Z}_{60}$ generated by 8 and 30.
   (d) Find all elements of the multiplicative group $\mathbf{Z}_{60}^*$.

2. (a) Define what it means for two groups to be *isomorphic*.
   (b) Consider the following set of rational numbers:
   $$A = \left\{ 2^a 3^b \;\middle|\; a, b \in \mathbf{Z} \right\}.$$
   Show that A is an abelian group under multiplication.
   (c) Show that A is isomorphic with $\mathbf{Z} \times \mathbf{Z}$.

3. (a) Define an *algebraic extension* of a field.
   (b) Show that the ring
   $$K = \mathbf{Z}_2[x]/(x^4 + x + 1)$$
   is a field, and determine its order.

(c) Is the multiplicative group K^* cyclic?

(d) Is the field extension $\mathbf{Z}_2 \leqslant K$ algebraic?

4. (a) Let N be a normal subgroup of a group G. Define the *factor group* G/N.

   (b) Consider the map
   $$\pi\colon \mathbf{R}^2 \to \mathbf{R}, \quad (x, y) \mapsto x + y.$$
   Show that π is a group homomorphism.

   (c) Find the kernel and image of π.

   (d) Describe the cosets in $\mathbf{R}^2/\operatorname{Ker}\pi$.

5. (a) Define *prime* and *maximal ideals* in a commutative, unital ring.

   (b) Consider the ring
   $$R = \mathbf{C}[x]/(x^3 - 1).$$
   Show that the ideal $(x - 1 + (x^3 - 1))$ is both prime and maximal.

   (c) Show that the ideal $(0 + (x^3 - 1))$ is neither prime nor maximal.

   (d) Does R contain an ideal which is prime, but not maximal?

   (e) Does R contain an ideal which is maximal, but not prime?

6. (a) Define what it means for a polynomial equation $p(x) = 0$ to be *soluble in radicals* over $\mathbf{Q}$.

   (b) Find the Galois group of the polynomial equation
   $$x^5 - x^4 - x + 1 = 0.$$

   (c) Is the equation soluble in radicals?

7. Let L be a *finite* commutative, unital ring of characteristic 3. Suppose that, for any inversible $u \neq 1$, the element $u - 1$ will be inversible as well.

   (a) Show that the map
   $$\varphi\colon L \to L, \quad x \mapsto x^3$$
   is a ring homomorphism.

   (b) Show that φ is a ring isomorphism.

   (c) Show that there exists a positive integer n such that $x^{3^n} = x$ for all $x \in L$.

   (d) Let now x be an arbitrary element of L. Show that $x^{3^n-1} + 1$ is its own inverse.

   (e) Show that L, in fact, must be a field.