ALGEBRAIC STRUCTURES

Xantcha

Examination 26th August 2014

Solutions. Complete solutions are required for each problem.

Marking. Each problem is worth 6 points.

- The marks 3, 4 and 5 correspond approximately to the scores 18, 25 and 32, respectively, distributed reasonably evenly among the three subdivisions Group Theory, Ring Theory and Field Theory.
- Also, in order to pass, a student should demonstrate some knowledge of the fundamental definitions of the course. Definitions should be written out formally, using complete sentences.

1. (a) Define a *group*.

   (b) Define a *subgroup*.

   (c) Let G be a group, and define a map

   $$\gamma\colon G \times G \to G, \qquad (x, y) \mapsto xy^{-1}.$$

   Shew that $\gamma(x, x) = 1$.

   (d) Shew that $\gamma(x, y)\gamma(y, x) = 1$.

   (e) Shew that $\gamma(\gamma(x, z), \gamma(y, z)) = \gamma(x, y)$.

2. (a) Let $F \leqslant E$ be fields. Define the *degree over F* of an algebraic element $\alpha \in E$.

   (b) Shew that $\alpha = \sqrt{2 + \sqrt{2}}$ is algebraic of degree 4 over $\mathbf{Q}$.

   (c) Find a field over which α becomes algebraic of degree 2.

3. (a) Define the *symmetric group on n symbols*, S_n.

(b) Explain, by means of (non-trivial) examples, how to perform multiplication and take inverses of permutations in S_n.

(c) Define what it means for a permutation in S_n to be *even or odd*.

(d) Give examples of even and odd permutations (and indicate why they are even and odd, respectively).

4. (a) Define a *group homomorphism*.

   (b) Shew that the map

$$\varphi\colon \mathbf{C} \to \mathbf{C}^*, \qquad z \mapsto e^z$$

   is a group homomorphism. ($\mathbf{C}^*$ denotes the same as $\mathbf{C}\setminus\{0\}$.)

   (c) Determine the image and kernel of φ.

   (d) What does the Fundamental Homomorphism Theorem say, when applied to φ?

5. (a) Starting from the concept of a ring (this need not be defined), define an *integral domain*. Give an example and a non-example.

   (b) Shew that a ring is an integral domain if and only if it is a subring of some field.

   (c) Let D be an integral domain and let $a, b \in D$. Shew that the principal ideals (a) and (b) are equal if and only if $b = au$ for some inversible element u.

6. Let R be a commutative, unital ring.

   (a) Define the concept of an *ideal* in R.

   (b) Let I and J be ideals of R. The *ideal sum* $I + J$ is defined by

$$I + J = \{\, x + y \mid x \in I,\ y \in J \,\}.$$

   Shew that $I + J$ is an ideal.

   (c) Shew that the rings $I + J$ and $I \times J$ are isomorphic under the hypothesis that $I \cap J = \{0\}$.

7. Let $F \leqslant E$ be a field extension.

   (a) Define what it means for this field extension to be *algebraic*.

   (b) Define what it means for this field extension to be *finite*.

   (c) Shew that $F \leqslant E$ is algebraic if and only if each ring R such that $F \leqslant R \leqslant E$ is a field.