Uppsala universitet
Matematiska institutionen
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Prov i matematik
Algebraic structures, 10hp
2014–12–15

Skrivtid: 8.00–13.00. Inga hjälpmedel förutom skrivdon. Lösningarna skall åtföljas av förklarande text. Varje uppgift ger maximalt 5 poäng.

1. (a) When is a subset H of a group G called a *subgroup* of G? Reproduce the definition!

(b) Show that the set of complex numbers $\mathrm{C}_4 = \{1, -1, i, -i\}$ is a subgroup of the multiplicative group $\mathbb{C} \setminus \{0\}$.

(c) Classify all groups of order 4.

(d) To which of the groups in (c) is C_4 isomorphic? Give reasons for your answer!

2. (a) Classify all abelian groups of order 243.

(b) Classify all groups of order 289.

(c) Quote the general results you used in your solutions to (a) and (b).

3. A permutation $\sigma \in \mathrm{S}_{12}$ is given as follows:

x	1	2	3	4	5	6	7	8	9	10	11	12
$\sigma(x)$	12	1	7	11	10	8	2	9	6	4	5	3

(a) What is the order of σ?

(b) What is meant by the *alternating subgroup* $\mathrm{A}_{12} < \mathrm{S}_{12}$? Reproduce the definition!

(c) Decide whether σ belongs to A_{12} or not. Motivate your answer.

4. (a) When is a ring R called a *domain*? Reproduce the definition!

(b) Which of the following three quotient rings

$$R_1 = \mathbb{Z}[X]/(X^2 + X + 1), \; R_2 = \mathbb{Q}[X]/(X^2 + 1), \; R_3 = \mathbb{C}[X, Y]/(XY)$$

is a domain, and which is not? Motivate your answer.

Please turn over!

5. (a) What is an *irreducible element* in a domain? Reproduce the definition!

(b) What is a *prime element* in a domain? Reproduce the definition!

(c) Show that the polynomial $XY - Z^2$ is irreducible in $\mathbb{C}[X, Y, Z]$.

(d) Is the polynomial $XY - Z^2$ prime in $\mathbb{C}[X, Y, Z]$? Motivate your answer.

6. Decide whether the real number $r = \sqrt[19]{17000}$ is rational or not. Motivate your answer.

7. Find the degrees of the field extensions $\mathbb{Q} \subset \mathbb{Q}(\sqrt{5})$, $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{5})$, and $\mathbb{Q} \subset \mathbb{Q}(\sqrt{5}, \sqrt[3]{5})$. Find even a $\mathbb{Q}$-basis in $\mathbb{Q}(\sqrt{5}, \sqrt[3]{5})$.

8. (a) What is meant by a *Galois extension*? Reproduce the definition!

(b) Let $\zeta = e^{\frac{2\pi}{5}i}$ and $E = \mathbb{Q}(\zeta)$. Show that $\mathbb{Q} \subset E$ is a finite Galois extension.

(c) Show that there is a unique intermediate field $\mathbb{Q} \subset I \subset E$ such that $\mathbb{Q} \neq I$ and $I \neq E$.

(d) Give reasons for the existence of a complex number $s \in E$ such that $\mathbb{Q}(s) = I$.

(e) Find a complex number $s \in E$ such that $\mathbb{Q}(s) = I$.

GOOD LUCK!