Uppsala universitet
Matematiska institutionen
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Kandidatprogrammet i matematik
Autumn term 2015

Prov i matematik
Algebraic structures, 10hp
2015–12–11

Skrivtid: 8.00–13.00. Inga hjälpmedel förutom skrivdon. Lösningarna skall åtföljas av förklarande text. Varje uppgift ger maximalt 5 poäng.

1. Let $\mathbb{R}^\iota$ and $\mathbb{C}^\iota$ be the unit groups of $\mathbb{R}$ and $\mathbb{C}$, respectively.

(a) Show that $\varphi : \mathbb{C}^\iota \to \mathbb{R}^\iota,\ \varphi(z) = |z|$ is a group morphism.

(b) For any $z \in \mathbb{C}^\iota$, describe the coset $z(\ker\varphi)$ geometrically as a subset of the complex plane.

(c) The set $\mathbb{R}_{>0}$ of all positive real numbers and the unit circle $\mathbb{S}^1$ are subgroups of $\mathbb{R}^\iota$ and $\mathbb{C}^\iota$, respectively. Prove that $\mathbb{C}^\iota/\mathbb{S}^1 \xrightarrow{\sim} \mathbb{R}_{>0}$.

2. Explain why the following assertions hold true:

(a) Every group of order 86 has a unique normal subgroup of index 2.

(b) Every group of order 86 is solvable.

(c) Every abelian group of order 86 is cyclic.

(d) Non-abelian groups of order 86 exist.

3. The permutation $\sigma \in S_9$ is given in two-line notation by

i	1	2	3	4	5	6	7	8	9
$\sigma(i)$	2	4	6	8	7	9	5	1	3

Find the cycle decomposition of σ, its cycle type, its order, and the cardinalities $|K(\sigma)|$ and $|C(\sigma)|$ of the conjugacy class and the centralizer of σ, respectively.

4. For each $i \in \{1, 2, 3\}$ determine all rings R having the property (P_i), given as follows:

(P_1) The identity $x + y = xy$ holds for all $x, y \in R$.

(P_2) There exists a ring morphism $\varphi : \{0\} \to R$.

(P_3) There exists a ring morphism $\varphi : R \to \{0\}$.

Please turn over!

5. Let ζ be the complex number $\zeta = \frac{1+i}{\sqrt{2}}$. Find the degree $d = [\mathbb{Q}(\zeta) : \mathbb{Q}]$, and find the rational coordinates of $\frac{1}{1+\zeta}$ in the $\mathbb{Q}$-basis $(1, \zeta, \ldots, \zeta^{d-1})$ of $\mathbb{Q}(\zeta)$.

6. Determine the degree $[\mathbb{C}(\alpha) : \mathbb{C}]$ for all $\alpha \in \mathrm{frac}(\mathbb{C}[X])$.

7. Let K be a field, and $f(X) \in K[X]$ a polynomial with coefficients in K.

(a) What is meant by a *splitting field* of $f(X)$? Reproduce the definition!

(b) Does a splitting field of $f(X)$ exist, and if so, in which sense is it unique? Reproduce the statement!

(c) Let E and F be splitting fields of $f(X)$. Suppose that all roots of $f(X)$ in E are simple. What can you say about the multiplicities of the roots of $f(X)$ in F? Prove your statement!

8. Let p be a prime natural number. Prove the following statements:

(a) The identity $x^p = x$ holds for all elements $x \in \mathbb{Z}_p$.

(b) The identity $(f(X))^p = f(X^p)$ holds for all polynomials $f(X) \in \mathbb{Z}_p[X]$.

(c) Every finite field extension $\mathbb{Z}_p \subset E$ is Galois.

GOOD LUCK!