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Prov i matematik
Algebraic structures, 10hp
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Skrivtid: 14:00–19:00. Inga hjälpmedel förutom skrivdon. Lösningarna skall åtföljas av förklarande text. Varje uppgift ger maximalt 5 poäng.

1. Let $\mathbb{C}^\iota$ and $\mathbb{S}^1$ be the multiplicative group of $\mathbb{C}$ and the unit circle in the complex plane, respectively.

(a) Show that the map $\psi : \mathbb{C}^\iota \to \mathbb{S}^1,\ \psi(z) = \frac{z}{|z|}$ is a group morphism.

(b) For any $z \in \mathbb{C}^\iota$, describe the coset $z(\ker\psi)$ geometrically, as a subset of the complex plane.

(c) Prove that $\mathbb{C}^\iota/\mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{S}^1$, where $\mathbb{R}_{>0}$ is the set of all positive real numbers.

2. (a) Determine all natural numbers $1 \leq n \leq 9$ such that a non-abelian group of order n exists.

(b) Classify all abelian groups of order less or equal to 9.

3. The permutation $\sigma \in S_{11}$ is given in two-line notation by

i	1	2	3	4	5	6	7	8	9	10	11
$\sigma(i)$	9	10	8	2	3	7	4	1	11	6	5

Find the cycle decomposition of σ, its cycle type, its order, and the cardinalities $|K(\sigma)|$ and $|C(\sigma)|$ of the conjugacy class and the centralizer of σ, respectively.

4. (a) Show that, for every ring R, there is a unique ring morphism $\varphi : \mathbb{Z} \to R$.

(b) The set $S = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ is a ring, with componentwise defined addition and multiplication. Determine $\ker\varphi$ and $\mathrm{im}\varphi$, for the unique ring morphism $\varphi : \mathbb{Z} \to S$.

(c) According to the Isomorphism Theorem for Rings, the ring morphism $\varphi : \mathbb{Z} \to S$ in (b) induces a ring isomorphism $\overline{\varphi} : A \to B$. Make the rings A and B explicit.

Please turn over!

5. (a) What is meant by a domain? Reproduce the definition!

(b) Prove that the ring $R = \mathbb{C}[X,Y]/(X^2+Y^2-1)$ is a domain.

6. Let K be a finite field of odd order q, and let K^ι be its multiplicative group. The set $K^\iota_{sq} = \{x^2 \mid x \in K^\iota\}$ of all squares in K^ι is a subgroup of K^ι.

(a) Determine the order of K^ι_{sq}, and the index of K^ι_{sq} in K^ι.

(b) Let $a \in K^\iota \setminus K^\iota_{sq}$. Show that $E_a = K[X]/(X^2 - a)$ is a finite field of order q^2.

(c) Let $a, b \in K^\iota \setminus K^\iota_{sq}$. Explain why the fields E_a and E_b, defined as in (b), are isomorphic.

7. Let $q = p^n$, where p is prime and $n \in \mathbb{N} \setminus \{0\}$. Let $\mathbb{F}_p$ and $\mathbb{F}_q$ be fields of order p and q, respectively. The field extension $\mathbb{F}_p \subset \mathbb{F}_q$ is Galois, with Galois group G.

(a) What is the order of G? Motivate your answer!

(b) Show that the map $\sigma : \mathbb{F}_q \to \mathbb{F}_q,\ \sigma(x) = x^p$, is an element in G.

(c) What is the order of σ? Motivate your answer!

(d) Use (a)–(c) to determine the structure of G.

8. Let $E = \mathbb{Q}(\zeta)$, where $\zeta = e^{\frac{2\pi}{7}i}$. Find a primitive element for each intermediate field $\mathbb{Q} \subset I \subset E$, such that $\mathbb{Q} \neq I$ and $I \neq E$.

Good luck!