Uppsala universitet
Matematiska institutionen
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Prov i matematik
Algebraic structures, 10hp
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Skrivtid: 8:00–13:00. Inga hjälpmedel förutom skrivdon. Lösningarna skall åtföljas av förklarande text. Varje uppgift ger maximalt 5 poäng.

1. Let $\mathbb{S}^1$ be the unit circle in the complex plane.

(a) Show that $\mathbb{S}^1$ is a group under complex multiplication.

(b) Show that $\mathbb{Z}^2$ is an additive normal subgroup of $\mathbb{R}^2$.

(c) Prove that the quotient group $\mathbb{R}^2/\mathbb{Z}^2$ is isomorphic to the product group $\mathbb{S}^1 \times \mathbb{S}^1$.

2. Classify all finite abelian groups G of order $12 \leq |G| \leq 16$.

3. (a) Prove that every group of order 111 has a unique normal subgroup of index 3.

(b) Quote the Theorem of Feit and Thompson, regarding solvable groups.

(c) Prove, without invoking Feit-Thompson's Theorem, that every group of order 111 is solvable.

(d) Show that every abelian group of order 111 is cyclic.

4. The permutation $\sigma \in S_{12}$ is given in two-line notation by

i	1	2	3	4	5	6	7	8	9	10	11	12
$\sigma(i)$	9	10	8	2	3	7	4	1	11	6	12	5

Find the cycle decomposition of σ, its cycle type, its order, and the cardinalities $|K(\sigma)|$ and $|C(\sigma)|$ of the conjugacy class and the centralizer of σ, respectively.

5. (a) Let R be any ring. What is meant by a (two-sided) R-ideal? Reproduce the definition!

(b) Determine all R-ideals for the ring $R = \mathbb{R}^{2\times 2}$.

Please turn over!

6. Let p be a prime number, and let $\mathbb{F}_p$ and $\mathbb{F}_{p^3}$ be finite fields of order p and p^3, respectively.

(a) Show that there is an injective ring morphism $\varphi : \mathbb{F}_p \to \mathbb{F}_{p^3}$.

(b) Prove that every element $a \in \mathbb{F}_p$ has a third root in $\mathbb{F}_{p^3}$.

7. Let $k \subset \ell$ be a field extension of degree 2, such that $\mathrm{char}(k) \neq 2$.

(a) Prove that $k \subset \ell$ is a Galois extension, whose Galois group G has order 2.

(b) Let $\sigma \in G \setminus \{\mathbb{I}\}$. Prove that the subset $I = \{x \in \ell \mid \sigma(x) = -x\} \subset \ell$ is a 1-dimensional k-linear subspace of ℓ, such that $k + I = \ell$ and $k \cap I = \{0\}$.

(c) Show that $I = \{x \in \ell \mid x^2 \in k\} \setminus (k \setminus \{0\})$.

8. Let $\zeta = e^{\frac{2\pi}{37}i}$. Determine all intermediate fields $\mathbb{Q} \subset I \subset \mathbb{Q}(\zeta)$.

Good luck!