MATEMATISKA INSTITUTIONEN
UPPSALA UNIVERSITET
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Tentamensskrivning
Algebraiska strukturer 10 hp
2016-12-15 kl 08-13

- *The usual means are allowed: pen, pencil, eraser, ruler and compass.*
- *The scores 20p, 27p and 34p correspond to the grades 3, 4 and 5 respectively.*
- *Complete solutions, where all the steps are clearly explained, are required.*

A ring is assumed to have a unity. A ring homomorphism $R \to S$ maps 1_R to 1_S.

(1) Decide if the following statements are true or false. Correct answer gives 0,5 p, wrong answer -0,5 p and no answer 0 p. You can get between 0 p och 5 p on this question.
  (a) Any prime element in an integral domain R is irreducible in R.
  (b) If a prime number p divides the order of the group G, then there is a subgroup in G of order p.
  (c) $|A_n| = |S_n|/2$.
  (d) The ring $\mathbb{Z}_{11}$ has no zerodivisors.
  (e) The group S_n is the only group of permutations on the set $\{1, \ldots, n\}$.
  (f) The group A_4 is abelian.
  (g) Any group homomorphism is a ring homomorphism.
  (h) A subgroup H of G is called normal if $gHg = H$ for all $g \in G$.
  (i) The identity permutation is an odd permutation.
  (j) The polynomial ring $R[x]$ over a commutative ring R is an integral domain.

(2) (a) Classify all abelian groups of order ≤ 15.
  (b) For each $4 \leq n \leq 15$, give an example of a non-abelian group of this order, if it exists. If there is no non-abelian group of order n for some n, you do not need to prove that.

(3) Prove or give a counterexample to the following statements.
  (a) Let S be a set with an associative binary operation on it. Assume that there is a left identity element $e \in S$ and that for every $x \in S$ there is a right inverse in S with respect to e. Then S is a group.
  (b) Let $\phi : R \to S$ a ring homomorphism. If $J \subset S$ is an ideal of S, then $\phi^{-1}(J)$ is an ideal of R.

(4) Consider the element $\sigma = (1)(12\ \ 5\ \ 2\ \ 6)(8\ \ 9\ \ 10\ \ 11)(3)(4\ \ 7) \in S_{12}$.
  (a) What is $ord(\sigma)$?
  (b) Determine the size of the conjugacy class of σ.
  (c) Determine the size of the centralizer of σ.
  (d) Is σ an even or odd permutation?

(5) Let $f(x) = x^4 + x^3 - 3x^2 + 3x + 3$ and $g(x) = x^5 - 2x^4 - x^3 + 3x^2 + 2x - 1$ be two polynomials in $\mathbb{Z}_7[x]$. Find the greatest common divisor of $f(x)$ and $g(x)$.

(6) Let $S = C^0(\mathbb{R})$ be the set of continuous functions on $\mathbb{R}$.
  (a) Show that S is a ring under the operations addition and (the ordinary) multiplication of functions.
  (b) Is the set consisting of all constant functions in S a subring?
  (c) Is the set consisting of all constant functions in S an ideal?
  (d) For which real numbers c is the set $\{f(x) \in S \mid f(2) = c\}$ an ideal?

TURN OVER, PLEASE!

(7) Consider the field extensions $\mathbb{Q} \subset \mathbb{Q}(\sqrt{3}, \sqrt{5}) = E$.

   (a) What is the degree of the extension? Give a $\mathbb{Q}$-basis of E.

   (b) There is some α such that $E = \mathbb{Q}(\alpha)$. What is $Irr(\alpha : \mathbb{Q})$?

(8) Construct a field of 8 elements.

   Hint. There are at least two ways to do that starting from the field $\mathbb{Z}_2$.

LYCKA TILL!