MATEMATISKA INSTITUTIONEN
UPPSALA UNIVERSITET
Examinator: Veronica Crispin Quiñonez

Tentamensskrivning
Algebraiska strukturer 10 hp
2017-04-18 kl 08-13

- *The usual means are allowed: pen, pencil, eraser, ruler and compass.*
- *Each problem is worth 5 ponts. The scores 20p, 27p and 34p correspond to the grades 3, 4 and 5 respectively.*
- *Complete solutions with all steps clearly explained, are required for problems 2-8.*

---

(1) State whether the following statements are true or false. Correct answer is 0,5 p, wrong answer $-0,5$ p, no answer 0 p. You can get minimum 0 p on this question.

(a) $\mathbb{Z}_n$ is a principal ideal domain for all $n \geq 2$.
(b) Any ideal is closed under multiplication.
(c) If two distinct prime numbers p and q divide the order of a group G, then there is a subgroup in G of order pq.
(d) Any field is an integral domain.
(e) The alternating group A_6 is solvable.
(f) Every finite field has p elements, where p is some prime number.
(g) The ring of quadratic matrices $M_6(\mathbb{C})$ contains zerodivisors.
(h) Every field homomorphism is injective.
(i) $\mathbb{Z}_3 \times \mathbb{Z}_3$ is a cyclic group.
(j) π is an algebraic number.

(2) Classify all abelian groups of order 2016.

(3) (a) Describe all normal non-trivial proper subgroups of S_4.
(b) Give a definition of a composition series of a group.
(c) Find a composition series of S_4.

(4) Prove the following statements.
(a) Let A and B be two groups, and let $\phi : A \to B$ be a group homomorphism. Show that $\phi(A)$ is a subgroup of B.
(b) Let R be a ring and $\mathfrak{m}$ an ideal in it. Then $R/\mathfrak{m}$ is a field if and only if $\mathfrak{m}$ is a maximal ideal.

(5) Consider the polynomial ring $\mathbb{R}[x, y]$. Let A be the ideal generated by the monomials x^3 and y^4, while B generated by the monomials x^4 and y^3, which we denote as $A = \langle x^3, y^4 \rangle$ and $B = \langle x^4, y^3 \rangle$ respectively. Give an irredundant generating set of the ideal product AB, that is, the set of all finite sums on the forms $\sum_i f(x,y)_i \cdot g(x,y)_i$, where $f(x,y)_i \in A$ and $g(x,y)_i \in B$ for all i.

(6) In this course all considered rings assumed to *unital*, that is, possessing a multiplicative identity. Moreover, a ring homomorphism $\phi : R \to S$ preserves the identity elements, that is, $\phi(1_R) = 1_S$.
(a) State the remaning conditions for a ring homomorphism.
(b) Let us now skip the condition on the identity elements being preserved. Determine in this case all ring homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_6$.
Hint. Such a homomorphism is determined by its action on 1.

(7) (a) Let K and L be subfields of a field F. Show that $K \cap L$ is a subfield of F.
(b) Give an example of F, K, L as in above so that $K \cup L$ is not a subfield of F.

(8) Let E be the splitting field in $\mathbb{C}$ of the polynomial $x^3 - 2$ over $\mathbb{Q}$. Determine the Galois group $\Gamma(E : \mathbb{Q})$.

LYCKA TILL!