MATEMATISKA INSTITUTIONEN
UPPSALA UNIVERSITET
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Lösningsförslag till tentamen
Algebraiska strukturer 10 hp
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(1) (a) T (b) T (c) F (d) T (e) F (T for $n \leq 4$) (f) F (T for p^n)
(g) T (h) T (i) F (j) F.

(2) $2016 = 2^5 \cdot 3^2 \cdot 7$. By The Fundamental thm of fin. gen. abelian groups (0,5 p) the non-isomorphic abelian groups are: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$,
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$, $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$,
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$, $\mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$,
$\mathbb{Z}_2 \times \mathbb{Z}_{16} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$, $\mathbb{Z}_{32} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$;
then just as many but with $\mathbb{Z}_9$ instead of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

(3) (a) The alternating subgroup is normal, here A_4 (1p). Moreover, there is the Klein group $K = \{(1), (12)(34), (13)(24), (14)(23)\}$ (1p).
(b) (1p) (c) For example, $(1) \triangleleft \langle (12)(34) \rangle \triangleleft K \triangleleft A_4 \triangleleft S_4$. (2p)

(4) (a) Standard exercise. (2,5p) (b) Proposition 4.6 in the course book. (2,5p)

(5) By definition any $f_i \in A$ and $g_i \in B$ can be written as $f_i = f_{i1}x^3 + f_{i2}y^4$ and $g_i = g_{i1}x^4 + g_{i2}y^3$ where $f_{ij}, g_{ij} \in \mathbb{R}[x, y]\}$. (1p)

Hence, an element in AB is a finite sum of the elements on the form $(f_{i1}x^3 + f_{i2}y^4) \cdot (g_{i1}x^4 + g_{i2}y^3) = f_{i1}g_{i1}x^7 + f_{i1}g_{i2}x^3y^3 + f_{i2}g_{i1}x^4y^4 + f_{i2}g_{i2}y^7 = f_{i1}g_{i1}x^7 + (f_{i1}g_{i2} + f_{i2}g_{i1}xy)x^3y^3 + f_{i2}g_{i2}y^7 = p_{i1}x^7 + p_{i2}x^3y^3 + p_{i3}y^7$ with $p_{ij} \in \mathbb{R}[x, y]$. As any element in AB is a sum of multiples of x^7, x^3y^3 and y^7, these three monomials form a generating set for AB. (3p)

Moreover, the set is irredundant as, for example, x^7 can never be expressed as $p_1x^3y^3 + p_2y^7$. The same goes for the other generators. (1p)

(6) (a) 0,5p per condition.
(b) For any $\phi(1_{12}) = m_6$ the additivity condition is fulfilled, and the map is well-defined as if $a \equiv_{12} b$, which is equivalent to $a = b + 12n$, then $\phi(a) = \phi(b + 12n) = bm + 12nm \equiv_6 bm = \phi(b)$ (0,5p).
For the multiplicativity condition to be fulfilled we must have $\phi(1_{12})\phi(1_{12}) = m \cdot m \equiv_6 m = \phi(1_{12})$. Going through all elements of $\mathbb{Z}_6$ we see that 0, 1, 3 and 4 fulfills $m^2 \equiv_6 m$. Thus, there are four ring homomoprhisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_6$ defined by sending the identity to 0, 1, 3 or 4. (4p)

(7) (a) Standard exercise.
(b) Both $\mathbb{R}$ and $\mathbb{Q}[i]$ are subfields of $\mathbb{C}$, but $\sqrt{2} + \frac{i}{2}$ does not belong to their union.

(8) See the example in Section V.5 in the course book or the solution to bonus exercies 3 from Nov 2016 posted on Studentportalen.