MATEMATISKA INSTITUTIONEN
UPPSALA UNIVERSITET
Examinator: Veronica Crispin Quiñonez

Tentamensskrivning
Algebraiska strukturer 10 hp
2017-08-15 kl 08-13

- *The usual means are allowed: pen, pencil, eraser, ruler and compass.*
- *Each problem is worth 5 ponts, unless it is stated otherwise. The scores 20p, 27p and 34p correspond to the grades 3, 4 and 5 respectively.*
- *Complete solutions, with all steps clearly explained, are required for problems 2-8.*

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(1) State whether the following statements are true or false. Correct answer is 0,5 p, wrong answer $-0,5$ p, no answer 0 p. You can get a minimum of 0 p on this question.
  (a) Every prime element is irreducible in an integral domain.
  (b) Any ideal is closed under addition.
  (c) If two distinct prime numbers p and q divide the order of a group G, then there is a subgroup in G of order pq.
  (d) Every integral domain is a field.
  (e) The alternating group A_8 is unsolvable.
  (f) Every finite field has p elements, where p is some prime number.
  (g) The ring of quadratic matrices $M_8(\mathbb{C})$ does not contain any zerodivisors.
  (h) The permutation $(1\ 3\ 4\ 2) \in S_4$ is even.
  (i) $\mathbb{Z}_8 \times \mathbb{Z}_{15}$ is a cyclic group.
  (j) $\sqrt[8]{15}$ is an algebraic number.

(2) Classify all groups of order ≤ 8, abelian as well as non-abelian.

(3) (a) State the definition of a (sub-)normal series of a group.
  (b) Find isomorphic refinements of the series $0 \trianglelefteq 27\mathbb{Z} \trianglelefteq 9\mathbb{Z} \trianglelefteq \mathbb{Z}$ and $0 \trianglelefteq 4\mathbb{Z} \trianglelefteq \mathbb{Z}$.
  (c) Write down all the factor groups in the refinements above. The order does not need to follow any of the refinements.

(4) Consider the permutation $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 1 & 3 & 4 & 6 & 5 & 2 & 9 & 10 & 7 \end{smallmatrix}\right)$. (2,5 p)
  (a) Write σ in cycle notation.
  (b) Determine the sign of σ.
  (c) Determine the order of σ in S_{10}.

(5) Prove that a field homomorphism is always injective. (2,5 p)

(6) Describe all the monomials of degree up to three in the ring $\mathbb{R}[x, y, z]/(y^3 - xz)$. Make sure the elements are written in the reduced form.

(7) (a) Let I be an ideal of a ring R and $\phi : R \to S$ a ring homomorphism. Show that if ϕ is surjective, then $\phi(I)$ is an ideal of S. Give a counterexampel if f is not surjective.
  (b) Give a counterexample when ϕ is not surjective.

(8) (a) Let K and L be subfields of a field F. Show that $K \cap L$ is a subfield of F.
  (b) Give an example of F, K, L as in above so that $K \cup L$ is not a subfield of F.

(9) (a) Show that a field of prime power order p^n is a splitting field over $\mathbb{Z}_p$ (or $\mathbb{F}_p$) of $x^{p^n} - x$.
  (b) Give the definition of a Galois extension.

LYCKA TILL!