MATEMATISKA INSTITUTIONEN	Tentamensskrivning
UPPSALA UNIVERSITET	Algebraiska strukturer 10 hp
Examinator: Veronica Crispin Quiñonez	2018-01-11 kl 08-13

- *The usual means are allowed: pen, pencil, eraser, ruler and compass.*
- *Each problem is worth 5 points. The scores 20p, 27p and 34p correspond to the grades 3, 4 and 5 respectively.*
- *Complete solutions, with all steps clearly explained, are required for problems 2-8.*

A ring is always assumed to have a unity, and a ring homomorphism $R \to S$ maps 1_R to 1_S, *unless* stated otherwise.

(1) State whether each of the following statements is true or false. Correct answer is worth 0,5p, wrong answer $-0,5$p, no answer 0p. You can get minimum 0p on this question. Solutions are not necessary, but not forbidden.

  (a) Every prime ideal is maximal.
  (b) The group A_6 is not solvable.
  (c) Every unique factorization domain is a principal ideal domain.
  (d) 1 is the only unit in $\mathbb{Z}$.
  (e) Taking square root of a number is a binary operation on $\mathbb{C}$.
  (f) Any ideal is closed under multiplication.
  (g) The permutation $(2\ 5\ 3\ 1) \in S_5$ is even.
  (h) π^2 is an algebraic number.
  (i) Every finite field has p elements, where p is a prime number.
  (j) It is possible to trisect the angle 90°.

(2) Consider the group $\langle \mathbb{Q}, + \rangle = \mathbb{Q}$.
  (a) Show that $\frac{1}{3^n}\mathbb{Z}$ is a subgroup of $\mathbb{Q}$ for every fixed $n \in \mathbb{N}$.
  (b) Show that $H = \bigcup_{n \geq 1} \frac{1}{3^n}\mathbb{Z} \leq \mathbb{Q}$.
  (c) Show that H is not finitely generated.

(3) Classify all abelian groups of order $20 \cdot 18$.

(4) (a) Show that $V_4 = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4.
  (b) Give the definition of a composition series of a group.
  (c) Find a composition series of S_4.

(5) (a) Give the definition of an integral domain.
  (b) Show that the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ is an integral domain.
  (c) Describe all units in $\mathbb{Z}[i]$. *Hint:* remember that $\mathbb{Z}[i] \subset \mathbb{C}$.

(6) (a) A ring is called *simple* if it has precisely two two-sided ideals. Determine whether each of the following three rings is simple or not: 0, $\mathbb{Z}$, any field K.
  (b) Explain why $\mathbb{Q}[x, y]$ and $\mathbb{Z}[x]$ are not principal ideal domains.

(7) (a) Show that $\alpha = \sqrt{2} + i\sqrt{3}$ is algebraic over $\mathbb{Q}$ and find its monic irreducible polynomial.
  (b) What is the degree of the extension $\mathbb{Q} \subset \mathbb{Q}[\alpha]$?
  (c) Write down a basis of $\mathbb{Q}[\alpha]$ as a vector space over $\mathbb{Q}$.

(8) Let E be the splitting field of $x^5 - 1$. Determine $\mathrm{Gal}(E/\mathbb{Q})$ and the corresponding fixed fields of the subgroups of the Galois group.

LYCKA TILL!