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Matematiska Institutionen
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Algebraiska strukturer
Kandidatprogrammet i matematik
HT 2020

Exam in Algebraic Structures
08-01-2021

Time: 14.00-19.00. You may look at your personal notes from the course, the lecture notes, and the course book. Please write your answers in English or in Swedish. Total is 40 points, of which you need 18 points for grade 3, 25 for grade 4, and 32 for grade 5.

1. (10 pt) Let S_3 be the set of permutations (i.e., bijections) on $\{1, 2, 3\}$.
   (a) Show that $(S_3, \circ)$ is a group, where the group operation $\circ$ is the composition of permutations.
   (b) Is S_3 abelian? Justify your answer.
   (c) Find all subgroups of S_3.
   (d) Find all quotient groups of S_3.
   (e) Is S_3 solvable? Justify your answer.
   (f) Show that S_3 is isomorphic to a quotient of the group G which has the following presentation by generators and relations:
   $$G = \langle b, c \mid bcb = cbc \rangle.$$

2. (5 pt) Classify all groups of order 33. Justify your answer.

3. (5 pt)
   (a) Does the element $X^3 - 3X + 9 \in \mathbb{Q}[X]$ generate a prime ideal in $\mathbb{Q}[X]$? Is the ideal maximal? Justify your answers.
   (b) Does the element $X^3 - 3X + 9 \in \mathbb{Z}[X]$ generate a prime ideal in $\mathbb{Z}[X]$? Is the ideal maximal? Justify your answers.
   (c) Does the element $X^3 - 3X + 9 \in \mathbb{Z}_7[X]$ generate a prime ideal in $\mathbb{Z}_7[X]$? Is the ideal maximal? Justify your answers.

4. (5 pt) Let R be a commutative ring and suppose the set $R \setminus R^{\times}$ is an ideal in R.
   (a) Prove that the characteristic of R is either zero or a power of a prime.
   (b) Given n, either zero or a prime power, can you find a ring R as above such that $\operatorname{char} R = n$? Justify your answer.

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5. (5 pt) Let K be a field, $f \in K[X]$ be an irreducible polynomial, and let $\alpha, \beta \in \overline{K} \setminus K$ be such that $f(\alpha) = f(\beta) = 0$.
   (a) Is the field extension $K \subseteq K(\alpha)$ finite? Justify your answer.
   (b) Show that $K(\alpha)$ is K-isomorphic to $K(\beta)$. (Construct a K-isomorphism $K(\alpha) \to K(\beta)$.)
   (c) Can the group $Aut_K(K(\alpha))$ be infinite? Can it be trivial? Justify your answers.

6. (5 pt) Let E be a finite field and let $K \subseteq E$ be a subfield. Prove or disprove: the extension $K \subseteq E$ is
   (a) normal;
   (b) separable.

7. (5 pt) Consider the polynomial $f = X^3 - 3 \in \mathbb{Q}[X]$ and let E be its splitting field (over $\mathbb{Q}$).
   (a) Prove that $\mathbb{Q} \subseteq E$ is Galois.
   (b) Compute the Galois group $Gal(E/\mathbb{Q})$.
   (c) Find all subfields $\mathbb{Q} \subseteq F \subseteq E$.
   (d) Which of these F are Galois over $\mathbb{Q}$ and what are their Galois groups $Gal(F/\mathbb{Q})$?