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Kandidatprogrammet i matematik
HT 2021

Exam in Algebraic Structures
07-01-2022

Time: 8.00-13.00. Notes, books or electronic devices are not allowed. Please write your answers in English or in Swedish. Total is 40 points, of which you need 18 points for grade 3, 25 for grade 4, and 32 for grade 5.

1. (10 pt) Let G be a group and let $H \leq G$ be a subgroup.
   (a) Consider the set of left cosets
   $$G/H = \{gH \mid g \in G\}.$$
   Give an example of a G-action on G/H. Justify your answer.
   (b) When is H said to be a normal subgroup?
   (c) Show that G/H is a group under the operation $(gH, g'H) \mapsto gg'H = (gg')H$ if and only if H is normal.
   (d) Show that, if $\phi : G \to L$ is a group homomorphism, then $\operatorname{Ker} \phi$ is a normal subgroup of G.
   (e) State the first isomorphism theorem for groups.

2. (5 pt) Give a presentation of the group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ by generators and relations.

3. (5 pt) A *Euclidean ring* is an integral domain R which admits a map
   $$R \setminus \{0\} \to \mathbb{Z}_{\geq 0}, \quad a \mapsto ||a||$$
   satisfying
   1. for $a, b \in R$, if b divides a, then $||b|| \leq ||a||$;
   2. for $a, b \in R$, if $b \neq 0$ does not divide a, then there is $q, r \in R$ with $||r|| < ||b||$ such that $a = bq + r$ holds.

   (a) Prove that a Euclidean ring is a principal ideal domain (PID).
   (b) Is $\mathbb{Q}[X, Y]$ a Euclidean ring? Justify your answer.

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4. (5 pt) Given a ring R and two rings S, T that contain R as a subring, an *R-homomorphism* $\phi : S \to T$ is a ring homomorphism such that $\phi(r) = r$ for $r \in R$.
   (a) Can you give two (different) examples of surjective $\mathbb{R}$-homomorphisms $\mathbb{R}[X] \to \mathbb{C}$? If yes, give the examples; if not, justify.
   (b) Can you give two (different) examples of injective $\mathbb{R}$-homomorphisms $\mathbb{R}[X] \to \mathbb{C}$. If yes, give the examples; if not, justify.

5. (5 pt) Consider the polynomial $f = X^{3^3} - X \in \mathbb{F}_3[X]$ and let E be its splitting field (over $\mathbb{F}_3$).
   (a) Is f irreducible?
   (b) Compute the group $\mathrm{Aut}_{\mathbb{F}_3}(E)$ of $\mathbb{F}_3$-automorphisms on E.
   (c) Find all subfields of E.

6. (5 pt) Show that, if $E \supset \mathbb{R}$ is finite Galois, then $[E : \mathbb{R}] = 2^r$ for some r. (Use the Galois theorem, the first Sylow theorem for $p = 2$ and the fact that an odd degree polynomial has a real root.)

7. (5 pt) Let $E \supset K$ be a field extension where $\operatorname{char} K = 0$. Recall that $E \supset K$ is said to be *solvable by radicals* if there exist field extensions

$$K = F_0 \subset F_1 \subset \cdots \subset F_l = E$$

   such that, for each i, we have $F_i = F_{i-1}(\alpha_i)$ with $\alpha_i^{m_i} \in F_{i-1}$ for some $m_i \in \mathbb{N}$.
   (a) Show that $\operatorname{char} E = 0$.
   (b) If E is the splitting field of some $f \in K[X]$ with $\deg f = 2$, then is E solvable over K? Justify your answer.
   (c) If $[E : K] = 2$, then is E solvable over K? Justify your answer.