Uppsala Universitet
Matematiska Institutionen
Hankyung Ko

Algebraiska strukturer
HT 2022

Exam in Algebraic Structures
03-01-2023

Notes, books or electronic devices are not allowed. Please write your answers in English or in Swedish. Total is 40 points, of which you need 18 points for grade 3, 25 for grade 4, and 32 for grade 5.

1. (5pt)
   (a) Let S be a set and let $\cdot : S \times S \to S$ be a function. When is $(S, \cdot)$ a group? That is, what is the definition of a group?
   (b) Define a normal subgroup of a group.

2. (5 pt) Recall that a composition series of a group G is a series of (proper) normal subgroups

$$\{e_G\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G,$$

   where each quotient group G_i/G_{i-1} is simple. Can you give two different composition series of the group G below? If yes, give the two composition series; if no, justify.
   (a) $G = \mathbb{Z}$.
   (b) $G = \mathbb{Z}/8\mathbb{Z}$.
   (c) $G = S_3 (= S_{\{1,2,3\}}$ in the notation of Problem 3).

3. (5 pt) Prove that every group is isomorphic to a subgroup of $S_X = (\{f : X \to X \mid f \text{ is a bijection}\}, \circ)$ for some set X. Here, $\circ$ denotes the composition of functions.

4. (5 pt) Let R be a(n integral) domain and let Q be the fraction field of R.
   (a) Prove or disprove: if R is finite then Q is finite.
   (b) Prove or disprove: the characteristic of R is equal to the characteristic of Q.

Turn to the next page!

5. (5 pt) Let R be a principal ideal domain (PID). Let (p) be a nonzero ideal in R. Show that the following conditions are equivalent:

- p is prime;
- p is irreducible;
- (p) is prime;
- (p) is maximal.

6. (5 pt) Let K be a field, let $f \in K[X]$ be irreducible, and let $\alpha, \beta \in \overline{K}$ be such that $f(\alpha) = f(\beta) = 0$. Prove or disprove:
   (a) We have $K(\alpha) = K(\beta)$.
   (b) There is a K-isomorphism between $K(\alpha)$ and $K(\beta)$.
   (c) There is a K-isomorphism between $K(\alpha)$ and $K(\beta + 1)$.

7. (5 pt) Consider the field extension $E = \mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}]$ of $\mathbb{Q}$. Let $G = Aut_{\mathbb{Q}}(E)$.
   (a) Is the extension $\mathbb{Q} \subset E$ simple?
   (b) Is the extension $\mathbb{Q} \subset E$ separable?
   (c) Is the extension $\mathbb{Q} \subset E$ normal?
   (d) Show that, for $\sigma \in G \setminus \{e_G\}$, we have $|\sigma| = 2$. (Hint: what are the roots of $X^2 - 2 \in \mathbb{Q}[X]$ in E?)

8. (5 pt) If $E \supset \mathbb{R}$ is finite, then $[E : \mathbb{R}] = 2^r$ for some r. Use this fact to prove that the field $\mathbb{C}$ is algebraically closed. (Hint: use the Galois theorem and Sylow's theorem.)

The exam ends here.