

Time: 8.00-13.00. Limits for the credits 3, 4, 5 are 18, 25 and 32 points, respectively. The solutions should be well motivated.

Permitted aids: The course book or copies thereof. Hand-written sheet of formulae. Pocket calculator. Dictionary. *No electronic device with internet connection.*

1. Let  $\{w_t\}$ ,  $t = 0, 1, 2, \dots$  be a Gaussian white noise process with  $\text{var}(w_t) = \sigma_w^2 = 2$  and let (5p)

$$x_t = 2 + 3t + 0.5w_t^2.$$

- (a) Calculate the mean and autocovariance function of  $x_t$  and state whether it is weakly stationary.  
(b) Do the same for  $\nabla x_t = x_t - x_{t-1}$ .

*Hint:* It may be used without proof that  $E(w_t^4) = 3\sigma_w^4$ .

2. For the ARMA( $p, q$ ) models below, where  $\{w_t\}$  are white noise processes, find  $p$  and  $q$  and determine whether they are causal and/or invertible. (6p)

- (a)  $x_t = w_t + 0.4w_{t-1}$   
(b)  $x_t = 0.3x_{t-1} + 0.4x_{t-2} + w_t + 0.5w_{t-1}$   
(c)  $x_t = 0.125x_{t-3} + w_t - 0.125w_{t-1}$   
(d)  $x_t = w_t + 0.8w_{t-1} - 0.2w_{t-2}$

3. Let  $\{w_t\}$  be a white noise process with variance  $\sigma_w^2$  and define the stationary process  $x_t$  through

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t.$$

Define the autocovariance function as  $\gamma(h) = \text{cov}(x_{t+h}, x_t)$ .

From observations  $x_1, x_2, \dots, x_{200}$ , we have obtained the estimated autocovariances  $\hat{\gamma}(0) = 2.0$ ,  $\hat{\gamma}(1) = 1.0$  and  $\hat{\gamma}(2) = 0.35$ .

- (a) Estimate the parameters  $\sigma_w^2$ ,  $\phi_1$  and  $\phi_2$  using the method of moments. (3p)  
(b) Calculate a 95% confidence interval for  $\phi_1$ . (3p)

4. Consider the process

$$x_t = -0.1x_{t-1} + 0.46x_{t-2} - 0.08x_{t-3} + w_t - 0.4w_{t-1},$$

where  $\{w_t\}$  is normally distributed white noise with variance  $\sigma_w^2 = 0.1$ . We observe  $x_t$  up to time  $t = 100$ , where the last five observations are  $x_{96} = 0.5$ ,  $x_{97} = 0.4$ ,  $x_{98} = 0.3$ ,  $x_{99} = 0.2$  and  $x_{100} = 0.1$ .

(a) Predict the values of  $x_{101}$  and  $x_{102}$ . (3p)

(b) Calculate 95% prediction intervals for  $x_{101}$  and  $x_{102}$ . (3p)

5. In the appendix, figures 1-3, three different empirical time series are plotted. Figure 1 gives the quarterly consumption of bio fuel in Swedish industry (unit: Tera Joule) for the years 2009-2021. In figure 2, we find the size of the Swedish population from 1860 to 2021. Finally, figure 3 gives the number of employed people in Sweden (unit: thousands), as monthly data from April 2005 until March 2022. All this data comes from Statistics Sweden.

Figures 4-8 depict five estimated spectral densities (non parametric with smoothing). Three of these correspond to the series in figures 1-3. Match three of the estimated spectral densities in figures 4-8 with the series in figures 1-3, and motivate your answers. (5p)

6. Consider the time series

$$x_t = 0.5x_{t-1} + w_t + 0.9w_{t-5},$$

where  $\{w_t\}$  is normally distributed white noise with variance  $\sigma_w^2 = 4$ .

(a) If this is written as a SARIMA( $p, d, q$ )  $\times$  ( $P, D, Q$ ) $_s$  model with only nonzero coefficients, what are  $p, d, q, P, D, Q, s$  here? (1p)

(b) For the frequency  $\omega = 0.2$ , calculate the spectral density of  $x_t$ . (2p)

(c) For the frequency  $\omega = 0.2$ , calculate the spectral density of  $y_t = \frac{1}{5}(x_t + x_{t-1} + x_{t-2} + x_{t-3} + x_{t-4})$ . (2p)

*Hint:* For exact calculation, you may use that  $\cos(a) = \cos(2\pi - a)$  for any  $a$ , and that  $\cos(0.4\pi) + \cos(0.8\pi) = -1/2$ .

(d) Compare your results in (b) and (c) and comment! (1p)

7. Let

$$x_t = \beta_0 + \beta_1 t + \phi x_{t-1} + w_t,$$

where  $\{w_t\}$  white noise and  $x_0 = 0$ . Suppose that  $|\phi| \leq 1$ .

(a) Prove that (2p)

$$x_t = (\beta_0 + \beta_1 t)(1 + \phi + \dots + \phi^{t-1}) - \beta_1 \{\phi + 2\phi^2 + \dots + (t-1)\phi^{t-1}\} + \sum_{j=0}^{t-1} \phi^j w_{t-j}.$$

(b) Prove that, if  $\phi = 1$ , then (1p)

$$x_t = \beta_0 t + \beta_1 \frac{t(t+1)}{2} + \sum_{j=0}^{t-1} w_{t-j}.$$

(c) Motivate why, as a unit root test for this model, it is reasonable to test  $H_0: (\beta_1, \phi) = (0, 1)$  vs  $H_1: \neg H_0$ . (3p)

GOOD LUCK!

## Appendix: figures

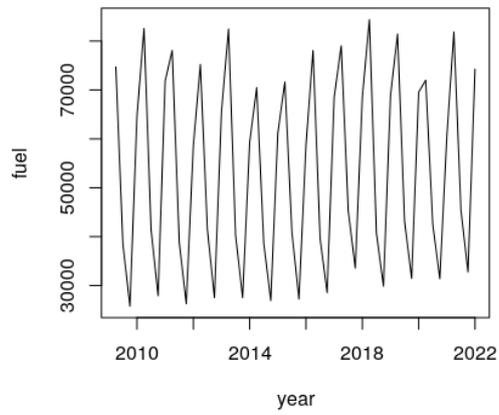


Figure 1: Quarterly industrial bio fuel consumption in Sweden.

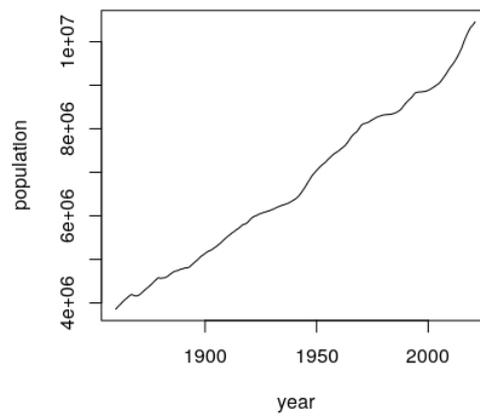


Figure 2: Population size in Sweden.

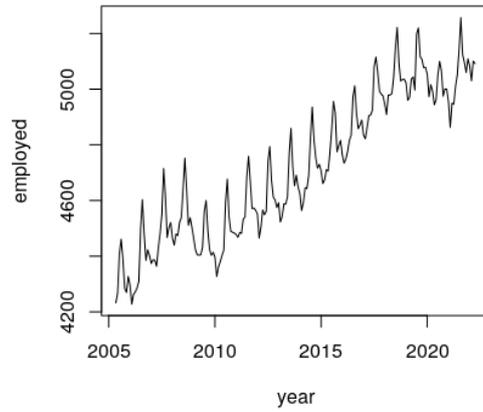


Figure 3: Monthly numbers of employed people in Sweden.

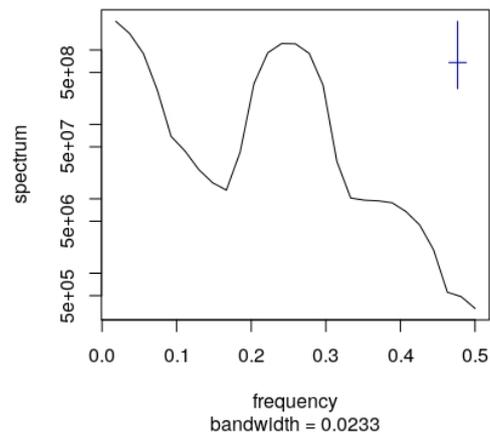


Figure 4: Spectral density, problem 5.

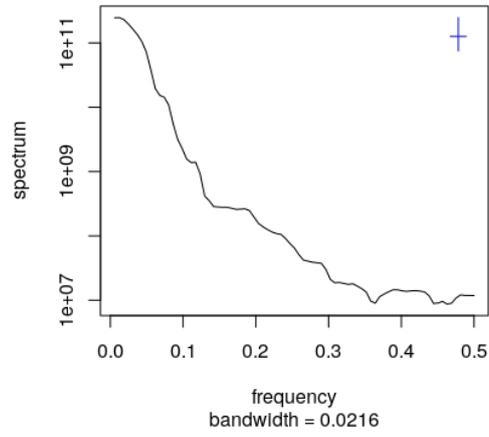


Figure 5: Spectral density, problem 5.

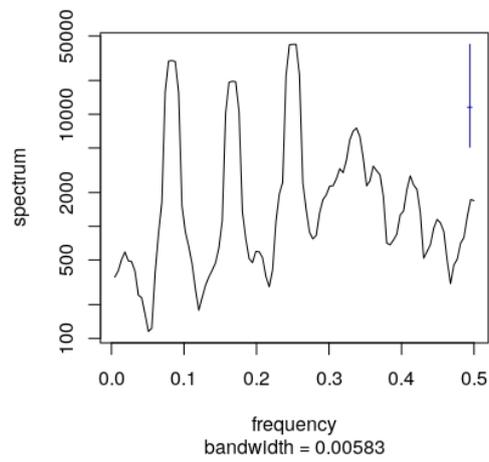


Figure 6: Spectral density, problem 5.

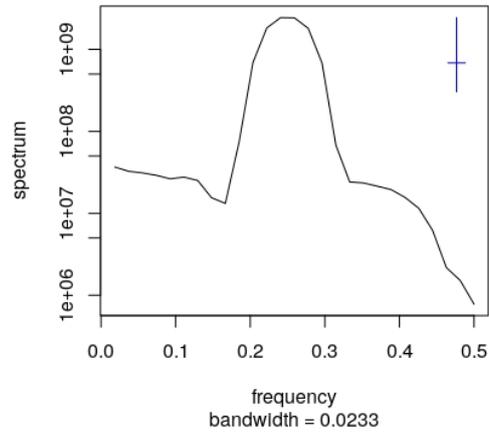


Figure 7: Spectral density, problem 5.

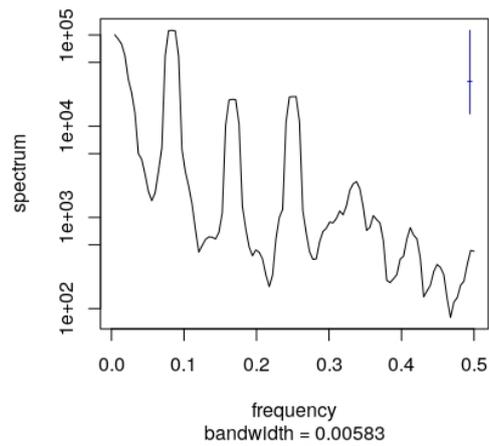


Figure 8: Spectral density, problem 5.