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Exam in Mathematical Statistics
Analysis of Time Series 1MS014
2022–05–25
Solutions

1. Let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = \sigma_w^2 = 2$ and let (5p)

$$x_t = 2 + 3t + 0.5w_t^2.$$

(a) Calculate the mean and autocovariance function of x_t and state whether it is weakly stationary.

Solution: Because $E(w_t) = 0$, we have $E(w_t^2) = \text{var}(w_t) = 2$ and the mean function is

$$\mu_t = E(x_t) = 2 + 3t + 0.5E(w_t^2) = 2 + 3t + 0.5 \cdot 2 = 3 + 3t.$$

The autocovariance function is (a constant is always uncorrelated to a random variable)

$$\begin{aligned}\gamma(t+h,t) &= \text{cov}(x_{t+h}, x_t) = \text{cov}\{2 + 3(t+h) + 0.5w_{t+h}^2, 2 + 3t + 0.5w_t^2\} \\ &= 0.25\text{cov}(w_{t+h}^2, w_t^2).\end{aligned}$$

Here, because of independence, $\text{cov}(w_{t+h}^2, w_t^2) = 0$ if $h \neq 0$. For $h = 0$, we have

$$\text{cov}(w_t^2, w_t^2) = E(w_t^4) - \{E(w_t^2)\}^2 = 3\sigma_w^4 - \sigma_w^4 = 2\sigma_w^4 = 8,$$

and so,

$$\gamma(t+h,t) = \begin{cases} 2 & \text{if } h = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since μ_t is a function of t, x_t is not weakly stationary.

(b) Do the same for $\nabla x_t = x_t - x_{t-1}$.

Solution: We get

$$\begin{aligned}\nabla x_t &= (2 + 3t + 0.5w_t^2) - \{2 + 3(t-1) + 0.5w_{t-1}\} \\ &= 3 + 0.5w_t^2 - 0.5w_{t-1}^2,\end{aligned}$$

from which we have the mean function

$$\mu_t = E(\nabla x_t) = 3 + 0.5E(w_t^2) - 0.5E(w_{t-1}^2) = 3 + 0.5 \cdot 2 - 0.5 \cdot 2 = 3,$$

and, similar to above, the autocovariance function

$$\begin{aligned}\gamma(t+h, t) &= \text{cov}(\nabla x_{t+h}, \nabla x_t) \\ &= \text{cov}\{3 + 0.5w_{t+h}^2 - 0.5w_{t+h-1}^2, 3 + 0.5w_t^2 - 0.5w_{t-1}^2\} \\ &= 0.25\{\text{cov}(w_{t+h}^2, w_t^2) - \text{cov}(w_{t+h}^2, w_{t-1}^2) - \text{cov}(w_{t+h-1}^2, w_t^2) \\ &+ \text{cov}(w_{t+h-1}^2, w_{t-1}^2)\} \\ &= 2(I\{h=0\} - I\{h=-1\} - I\{h=1\} + I\{h=0\}) \\ &= 4I\{h=0\} - 2I\{|h|=1\} = \begin{cases} 4 & \text{if } h = 0, \\ -2 & \text{if } |h| = 1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Hence, μ_t is constant and $\gamma(t+h, t)$ depends only on h, not on t. This means that ∇x_t is weakly stationary.

2. For the ARMA(p, q) models below, where $\{w_t\}$ are white noise processes, find p and q and determine whether they are causal and/or invertible. (6p)

   (a) $x_t = w_t + 0.4w_{t-1}$

   Solution: This is an MA(1) model, and so $p = 0$ and $q = 1$. It is causal, since all MA models are. We may write $x_t = \theta(B)w_t$, where $\theta(B) = 1 + 0.4B$. The root of $0 = \theta(z) = 1 + 0.4z$ is $z = -2.5$. Because $|z| = 2.5 > 1$, it is outside the complex unit circle. Hence, the model is invertible.

   (b) $x_t = 0.3x_{t-1} + 0.4x_{t-2} + w_t + 0.5w_{t-1}$

   Solution: Write $\phi(B)x_t = \theta(B)w_t$, where $\phi(B) = 1 - 0.3B - 0.4B^2$ and $\theta(B) = 1 + 0.5B$. At first sight, this seems to be ARMA(2,1), but we need to check that $\phi(z) = 0$ and $\theta(z) = 0$ have no common roots. The root of $\theta(z) = 1 + 0.5z = 0$ is $z = -2$, and in fact $\phi(-2) = 0$, so this is a common root.
   To find the roots of $\phi(z) = 1 - 0.3z - 0.4z^2 = 0$, we solve

$$z^2 + \frac{3}{4}z - \frac{5}{2} = 0,$$

   which has the solutions

$$z_{1,2} = -\frac{3}{8} \pm \sqrt{\frac{9}{64} + \frac{5}{2}} = \frac{-3 \pm 13}{8},$$

   i.e. $z_1 = -2$, $z_2 = 5/4 = 1.25$. Hence,

$$\phi(z) = -0.4(z + 2)(z - 5/4) = (1 + 0.5z)(1 - 0.8z),$$

   and we have

$$(1 + 0.5B)(1 - 0.8B)x_t = (1 + 0.5B)w_t,$$

   i.e. $(1 - 0.8B)x_t = w_t$ or $x_t = 0.8x_{t-1} + w_t$, an AR(1) model. Thus, $p = 1$ and $q = 0$.
   The model is invertible, since all AR models are. It is causal, because the root of $1 - 0.8z = 0$ is $z = 1.25 > 1$, hence outside the complex unit circle.

   (c) $x_t = 0.125x_{t-3} + w_t - 0.125w_{t-1}$

   Solution: This is ARMA(3,1), i.e. $p = 3$, $q = 1$, where $\phi(B)x_t = \theta(B)w_t$ with $\phi(B) = 1 - 0.125B^3$, $\theta(B) = 1 - 0.125B$. Solving $0 = \phi(z) = 1 - 0.125z^3$ gives $z = 2 > 1$, hence the model is causal. The solution to $0 = \theta(z) = 1 - 0.125B$ is $z = 8 > 1$, and so, the model is invertible.

(d) $x_t = w_t + 0.8w_{t-1} - 0.2w_{t-2}$

Solution: This is an MA(2) model, hence $p = 0$ and $q = 2$. We write $x_t = \theta(B)w_t$, where $\theta(z) = 1 + 0.8z - 0.2z^2$. The model is causal, since all MA models are.
The solutions to $0 = \theta(z)$ are the same as the solutions to $z^2 - 4z - 5 = 0$, which are given by

$$z_{1,2} = 2 \pm \sqrt{4+5} = 2 \pm 3,$$

i.e. $z_1 = -1$ and $z_2 = 5$. Hence, $|z_1| = 1$, i.e. this solution is not outside the complex unit circle.
This means that the model is not invertible.

3. Let $\{w_t\}$ be a white noise process with variance σ_w^2 and define the stationary process x_t through

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t.$$

Define the autocovariance function as $\gamma(h) = \mathrm{cov}(x_{t+h}, x_t)$.

From observations $x_1, x_2, ..., x_{200}$, we have obtained the estimated atocovariances $\hat{\gamma}(0) = 2.0$, $\hat{\gamma}(1) = 1.0$ and $\hat{\gamma}(2) = 0.35$.

(a) Estimate the parameters σ_w^2, ϕ_1 and ϕ_2 using the method of moments. (3p)

Solution: From the Yule-Walker equations, we get

$$\begin{aligned} \gamma(2) &= \phi_1\gamma(1) + \phi_2\gamma(0), && (1) \\ \gamma(1) &= \phi_1\gamma(0) + \phi_2\gamma(1), && (2) \\ \sigma_w^2 &= \gamma(0) - \phi_1\gamma(1) - \phi_2\gamma(2). && (3) \end{aligned}$$

Inserting the estimated autocovariances, and solving for the parameters in (1) and (2), we get the moment estimates $\hat{\phi}_1 = 0.55$ and $\hat{\phi}_2 = -0.1$. Then, insertion into (3) yields $\hat{\sigma}_w^2 = 1.485$.

(b) Calculate a 95% confidence interval for ϕ_1. (3p)

Solution: The asymptotic covariance matrix for the vector of estimators $(\hat{\phi}_1, \hat{\phi}_2)'$ is given by $n^{-1}\sigma_w^2\Gamma_p^{-1}$, where

$$\Gamma_p = \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}.$$

Inserting estimates, the upper left entry of the estimated Γ_p^{-1} is

$$\frac{\hat{\gamma}(0)}{\hat{\gamma}(0)^2 - \hat{\gamma}(1)^2} = \frac{2}{3},$$

which gives the estimated asymptotic variance of $\hat{\phi}_1$ as

$$\frac{1}{200} \cdot 1.485 \cdot \frac{2}{3},$$

and we get the 95% confidence interval

$$0.55 \pm 1.96\sqrt{\frac{1}{200} \cdot 1.485 \cdot \frac{2}{3}} = 0.55 \pm 0.14 = (0.41,\ 0.69).$$

4. Consider the process

$$x_t = -0.1x_{t-1} + 0.46x_{t-2} - 0.08x_{t-3} + w_t - 0.4w_{t-1},$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 0.1$. We observe x_t up to time $t = 100$, where the last five observations are $x_{96} = 0.5$, $x_{97} = 0.4$, $x_{98} = 0.3$, $x_{99} = 0.2$ and $x_{100} = 0.1$.

(a) Predict the values of x_{101} and x_{102}. (3p)

Solution: We will calculate truncated predictions by using the AR representation $\pi(B)x_t = w_t$. We have

$$(1 - 0.4B)w_t = (1 + 0.1B - 0.46B^2 + 0.08B^3)x_t,$$

which yields

$$\begin{aligned}\pi(B)(1 - 0.4B)w_t &= (1 + 0.1B - 0.46B^2 + 0.08B^3)\pi(B)x_t \\ &= (1 + 0.1B - 0.46B^2 + 0.08B^3)w_t,\end{aligned}$$

Hence, with $\pi(z) = 1 + \pi_1 z + \pi_2 z^2 + ...$, we need to solve

$$(1 + \pi_1 z + \pi_2 z^2 + ...)(1 - 0.4z) = 1 + 0.1z - 0.46z^2 + 0.08z^3,$$

i.e.

$$1 + (\pi_1 - 0.4)z + (\pi_2 - 0.4\pi_1)z^2 + ... = 1 + 0.1z - 0.46z^2 + 0.08z^3,$$

which yields

$$\begin{aligned}\pi_1 &= 0.4 + 0.1 = 0.5, \\ \pi_2 &= 0.4\pi_1 - 0.46 = -0.26, \\ \pi_3 &= 0.4\pi_2 + 0.08 = -0.024, \\ \pi_4 &= 0.4\pi_3 = -0.0096, \\ \pi_5 &= 0.4\pi_4 = -0.00384, \\ \pi_6 &= 0.4\pi_5 = -0.001536.\end{aligned}$$

The truncated predictions become

$$\begin{aligned}\tilde{x}_{101} &= -\pi_1 x_{100} - \pi_2 x_{99} - ... \\ &\approx -0.5 \cdot 0.1 + 0.26 \cdot 0.2 + 0.024 \cdot 0.3 + 0.0096 \cdot 0.4 + 0.00384 \cdot 0.5 \\ &= 0.01496\end{aligned}$$

and

$$\begin{aligned}\tilde{x}_{102} &= -\pi_1 \tilde{x}_{101} - \pi_2 x_{100} - \pi_3 x_{99} - ... \\ &\approx -0.5 \cdot 0.01496 + 0.26 \cdot 0.1 + 0.024 \cdot 0.2 + 0.0096 \cdot 0.3 \\ &\quad + 0.00384 \cdot 0.4 + 0.001536 \cdot 0.5 \\ &= 0.028504.\end{aligned}$$

(b) Calculate 95% prediction intervals for x_{101} and x_{102} . (3p)

Solution: The mean square predicion error m steps ahead is given by $\sigma_w^2 \sum_{j=0}^{m-1} \psi_j^2$, where the ψ_j are the coefficients in the MA representation, with $\psi_0 = 1$. We only need to find ψ_1. To this end, $x_t = \psi(B)w_t$ implies

$$\begin{aligned}\psi(B)(1 + 0.1B - 0.46B^2 + 0.08B^3)x_t &= (1 - 0.4B)\psi(B)w_t \\ &= (1 - 0.4B)x_t,\end{aligned}$$

so that with $\psi(z) = 1 + \psi_1 z + ...$, we have

$$(1 + \psi_1 z + ...)(1 + 0.1z - 0.46z^2 + 0.08z^3) = 1 - 0.4z,$$

implying $\psi_1 + 0.1 = -0.4$, i.e. $\psi_1 = -0.5$.
With $\sigma_w^2 = 0.1$, this gives the 95% prediction interval for x_{101} as

$$0.01496 \pm 1.96\sqrt{0.1} = 0.015 \pm 0.620 = (-0.605,\ 0.635),$$

and for x_{102}, we find the corresponding interval

$$0.028504 \pm 1.96\sqrt{0.1(1 + 0.5^2)} = 0.0285 \pm 0.6930 = (-0.664,\ 0.722).$$

5. In the appendix, figures 1-3, three different empirical time series are plotted. Figure 1 gives the quarterly consumption of bio fuel in Swedish industry (unit: Tera Joule) for the years 2009-2021. In figure 2, we find the size of the Swedish population from 1860 to 2021. Finally, figure 3 gives the number of employed people in Sweden (unit: thousands), as monthly data from April 2005 until March 2022. All this data comes from Statistics Sweden.

   Figures 4-8 depict five estimated spectral densities (non parametric with smoothing). Three of these correspond to the series in figures 1-3. Match three of the estimated spectral densities in figures 4-8 with the series in figures 1-3, and motivate your answers. (5p)

   Solution: Figure 1 depicts quarterly data, and as expected, we spot a season length of four, corresponding to a frequency of $1/4 = 0.25$. The series does not have much of a trend, so we expect a spectral density with a peak at 0.25 and not much more. Figures 4 and 7 both have peaks at 0.25, but the spectral density is lower close to zero in figure 7, whereas in figure 4, it is quite high there. So we settle for figure 7 as the best match.

   In figure 2, we see a clear trend and not much more, so the only structure of the corresponding spectral density should be that it is high at low frequencies. This is what we find in figure 5.

   The time series of figure 3 is monthly, and from the graph we can also see that high frequencies at multiples of $1/12 \approx 0.08$ are expected. We also see a clear increasing trend, so we would expect to see a spectral density that is large for small frequencies. In figures 6 and 8, we have spectral densities with peaks at multiples of $1/12$, but it is only the one in figure 8 that is high for low frequencies, so this is the one we choose.

   To conclude, we have the matches
   Figure 1- Figure 7,
   Figure 2- Figure 5,
   Figure 3- Figure 8.

6. Consider the time series

$$x_t = 0.5x_{t-1} + w_t + 0.9w_{t-5},$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 4$.

(a) If this is written as a SARIMA$(p, d, q) \times (P, D, Q)_s$ model with only nonzero coefficients, what are p, d, q, P, D, Q, s here? (1p)

Solution: We may write the model as

$$(1 - 0.5B)x_t = (1 + 0.9B^5)w_t,$$

and this may be seen as a SARIMA model with $p = 1$, $d = 0$, $q = 0$, $P = 0$, $D = 0$, $Q = 1$ and $s = 5$.

(b) For the frequency $\omega = 0.2$, calculate the spectral density of x_t. (2p)

Solution: Writing $\phi(B)x_t = \theta(B)w_t$ with $\phi(B) = 1 - 0.5B$ and $\theta(B) = 1 + 0.9B^5$, we get the spectral density

$$f_x(\omega) = \sigma_w^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2},$$

where in our case $\sigma_w^2 = 4$,

$$\begin{aligned} |\theta(e^{-2\pi i\omega})|^2 &= |1 + 0.9(e^{-2\pi i\omega})^5|^2 = |1 + 0.9e^{-10\pi i\omega}|^2 \\ &= (1 + 0.9e^{-10\pi i\omega})(1 + 0.9e^{10\pi i\omega}) \\ &= 1.81 + 0.9(e^{10\pi i\omega} + e^{-10\pi i\omega}) = 1.81 + 1.8\cos(10\pi\omega), \end{aligned}$$

and

$$\begin{aligned} |\phi(e^{-2\pi i\omega})|^2 &= |1 - 0.5e^{-2\pi i\omega}|^2 = (1 - 0.5e^{-2\pi i\omega})(1 - 0.5e^{2\pi i\omega}) \\ &= 1.25 - 0.5(e^{2\pi i\omega} + e^{-2\pi i\omega}) = 1.25 - \cos(2\pi\omega), \end{aligned}$$

so that

$$f_x(\omega) = 4 \cdot \frac{1.81 + 1.8\cos(10\pi\omega)}{1.25 - \cos(2\pi\omega)}.$$

Hence,

$$f_x(0.2) = 4 \cdot \frac{1.81 + 1.8\cos(2\pi)}{1.25 - \cos(0.4\pi)} \approx 4 \cdot \frac{3.61}{0.941} \approx 15.3.$$

(c) For the frequency $\omega = 0.2$, calculate the spectral density of $y_t = \frac{1}{5}(x_t + x_{t-1} + x_{t-2} + x_{t-3} + x_{t-4})$. (2p)

Hint: For exact calculation, you may use that $\cos(a) = \cos(2\pi - a)$ for any a, and that $\cos(0.4\pi) + \cos(0.8\pi) = -1/2$.

Solution: We have $y_t = \sum_j a_j x_{t-j}$, with the filter coefficients $a_j = 1/5$ for $j = 0, 1, 2, 3, 4$, and $a_j = 0$ otherwise.
We make use of the formula

$$f_y(\omega) = |A(\omega)|^2 f_x(\omega),$$

where

$$A(\omega) = \sum_j a_j e^{-2\pi i \omega j} = \frac{1}{5}\left(1 + e^{-2\pi i\omega} + e^{-4\pi i\omega} + e^{-6\pi i\omega} + e^{-8\pi i\omega}\right).$$

It follows in the usual manner that

$$|A(\omega)|^2 = \frac{1}{25}\left\{5 + 8\cos(2\pi\omega) + 6\cos(4\pi\omega) + 4\cos(6\pi\omega) + 2\cos(8\pi\omega)\right\},$$

which yields, using the hint,

$$\begin{aligned} |A(0.2)|^2 &= \frac{1}{25}\left\{5 + 8\cos(0.4\pi) + 6\cos(0.8\pi) + 4\cos(1.2\pi) + 2\cos(1.6\pi)\right\} \\ &= \frac{1}{25}\left\{5 + (8+2)\cos(0.4\pi) + (6+4)\cos(0.8\pi)\right\} = 0. \end{aligned}$$

Hence, $f_y(0.2) = |A(0.2)|^2 f_x(0.2) = 0$.

(d) Compare your results in (b) and (c) and comment! (1p)

Solution: We have a seasonal process x_t with period 5. When we take an even moving average over five consequtive values of the series, the seasonality is killed. This is why the peak at the frequency $\omega = 1/5$ disappears after this kind of filtering.

7. Let

$$x_t = \beta_0 + \beta_1 t + \phi x_{t-1} + w_t,$$

where $\{w_t\}$ white noise and $x_0 = 0$. Suppose that $|\phi| \leq 1$.

(a) Prove that (2p)

$$x_t = (\beta_0 + \beta_1 t)(1 + \phi + ... + \phi^{t-1})$$
$$- \beta_1\{\phi + 2\phi^2 + ... + (t-1)\phi^{t-1}\} + \sum_{j=0}^{t-1} \phi^j w_{t-j}.$$

Solution: Recursion gives

$$\begin{aligned}
x_t &= \beta_0 + \beta_1 t + \phi\{\beta_0 + \beta_1(t-1) + \phi x_{t-2} + w_{t-1}\} + w_t \\
&= (\beta_0 + \beta_1 t)(1+\phi) - \beta_1\phi + \phi^2 x_{t-2} + \phi w_{t-1} + w_t \\
&= (\beta_0 + \beta_1 t)(1+\phi) - \beta_1\phi \\
&+ \phi^2\{\beta_0 + \beta_1(t-2) + \phi x_{t-3} + w_{t-2}\} + \phi w_{t-1} + w_t \\
&= (\beta_0 + \beta_1 t)(1+\phi+\phi^2) - \beta_1(\phi + 2\phi^2) + \phi^3 x_{t-3} + \phi^2 w_{t-2} + \phi w_{t-1} + w_t \\
&= ... \\
&= (\beta_0 + \beta_1 t)(1 + \phi + ... + \phi^{t-1}) \\
&- \beta_1\{\phi + 2\phi^2 + ... + (t-1)\phi^{t-1}\} + \sum_{j=0}^{t-1} \phi^j w_{t-j}.
\end{aligned}$$

This may also be proved by induction.

(b) Prove that, if $\phi = 1$, then (1p)

$$x_t = \beta_0 t + \beta_1 \frac{t(t+1)}{2} + \sum_{j=0}^{t-1} w_{t-j}.$$

Solution: Inserting $\phi = 1$ in (a), we get

$$\begin{aligned}
x_t &= (\beta_0 + \beta_1 t) \cdot t - \beta_1\{1 + 2 + ... + (t-1)\} + \sum_{j=0}^{t-1} w_{t-j} \\
&= \beta_0 t + \beta_1 \left\{ t^2 - \frac{t(t-1)}{2} \right\} + \sum_{j=0}^{t-1} w_{t-j} \\
&= \beta_0 t + \beta_1 \frac{t(t+1)}{2} + \sum_{j=0}^{t-1} w_{t-j}.
\end{aligned}$$

(c) Motivate why, as a unit root test for this model, it is reasonable to test H_0: $(\beta_1, \phi) = (0, 1)$ vs H_1: $\neg H_0$. (3p)

Solution: For $\phi = 1$, the model is non stationary, and as is seen in (b), it has a quadratic trend. From (a) we find that for $|\phi| < 1$, the model is stationary since the stochastic part is a sum of "past" white noise terms with coefficients that decay exponentially fast. For a similar reason, the trend is linear.

Hence, without restricting β_1 under H_0, we test a non stationary model with quadratic trend vs a stationary model with linear trend. It is more natural to have a linear trend under both hypotheses, and that is what we accomplish by the restriction $\beta_1 = 0$ under H_0.

Appendix: figures

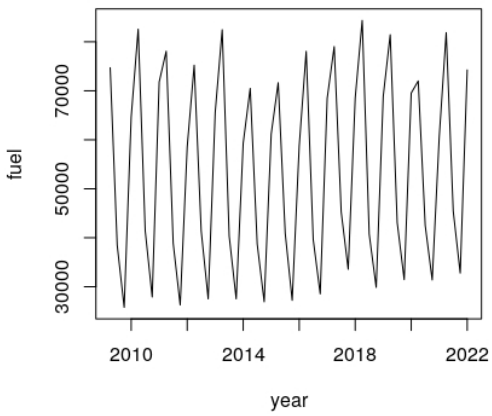

Figure 1: Quarterly industrial bio fuel consumption in Sweden.

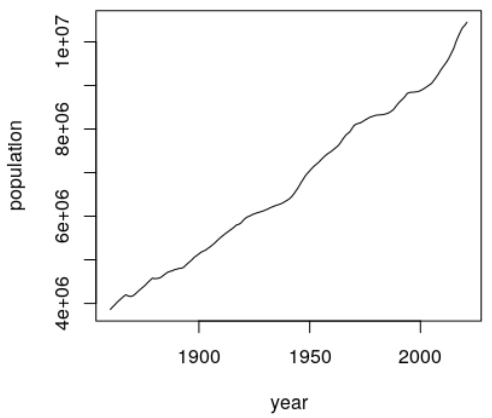

Figure 2: Population size in Sweden.

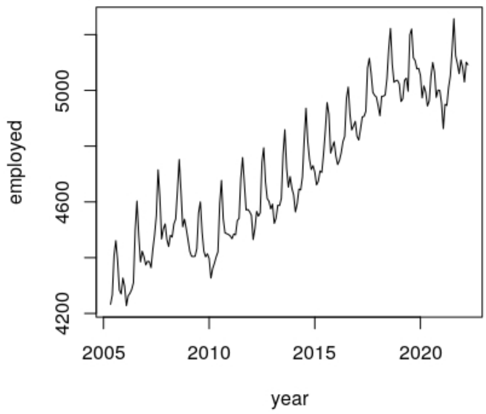

Figure 3: Monthly numbers of employed people in Sweden.

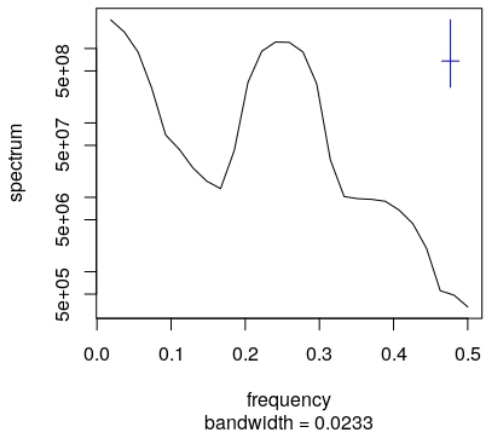

Figure 4: Spectral density, problem 5.

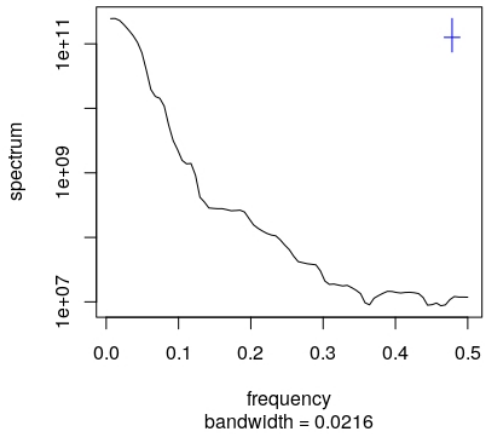

Figure 5: Spectral density, problem 5.

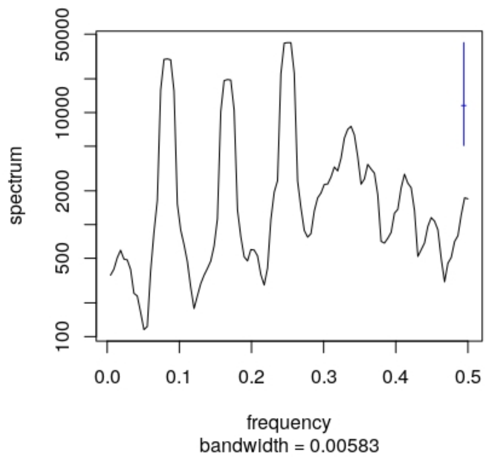

Figure 6: Spectral density, problem 5.

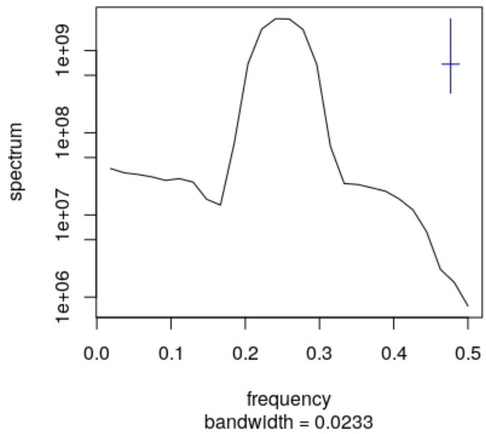

Figure 7: Spectral density, problem 5.

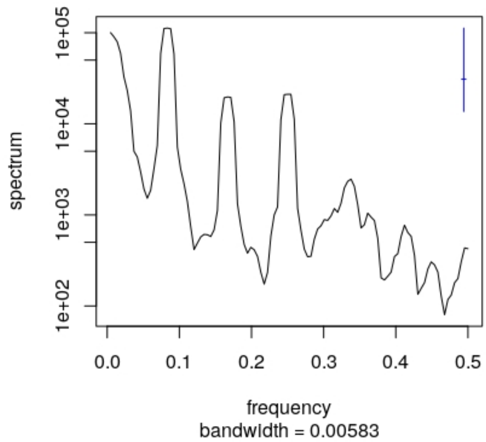

Figure 8: Spectral density, problem 5.