UPPSALA UNIVERSITY
Department of Mathematics
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Exam in Mathematical Statistics
Analysis of Time Series 1MS014
2023–05–30

Time: 8.00-13.00. Limits for the credits 3, 4, 5 are 18, 25 and 32 points, respectively, including bonus points. The solutions should be well motivated.

Permitted aids: The course book or copies thereof. Hand-written sheet of formulae. Pocket calculator. Dictionary. *No electronic device with internet connection.*

1. Let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 4$ and let
$$x_t = 3 + 0.5w_t^2 + 0.3w_{t-1}^2.$$
Calculate the mean and autocovariance function of x_t and state whether it is weakly stationary. (5p)

2. For the ARMA(p, q) models below, where $\{w_t\}$ are white noise processes, find p and q and determine whether they are causal and/or invertible. (6p)
   (a) $x_t = 0.8x_{t-1} + w_t$
   (b) $x_t = w_t + 0.4w_{t-1} + 0.05w_{t-2}$
   (c) $x_t = 0.5x_{t-1} + 0.5x_{t-2} + w_t - w_{t-1}$
   (d) $x_t = 0.6x_{t-1} + 0.4x_{t-2} + w_t + w_{t-4}$

3. Let $\{w_t\}$ be a white noise process with variance $\sigma_w^2 = 1$ and define x_t through
$$x_t = 0.5x_{t-4} + w_t + 0.5w_{t-1}.$$
Calculate the autocorrelation function $\rho(h)$ for $h = 1, 2, 3, 4$. (5p)

4. Consider the process
$$x_t = -0.4x_{t-1} + w_t - 0.7w_{t-1} + 0.1w_{t-2}$$
where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 0.1$. We observe x_t up to time $t = 100$, where the last four observations are $x_{97} = -0.1$, $x_{98} = -0.2$, $x_{99} = -0.2$ and $x_{100} = -0.1$.
   (a) Predict the values of x_{101} and x_{102}. Approximations are permitted. (4p)
   (b) Calculate 95% prediction intervals for x_{101} and x_{102}. (3p)

5. A time series $\{x_t\}$ follows the model

$$x_t = 0.2x_{t-1} + w_t,$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 1$. This series is used as input for constructing

$$y_t = 0.8y_{t-1} + x_t,$$

for $t = 0, \pm 1, \pm 2, ...$

(a) Calculate the spectral densitiy of x_t at the frequencies $\omega = 0.1$ and $\omega = 0.4$. (2p)

(b) Calculate the spectral densitiy of y_t at the frequencies $\omega = 0.1$ and $\omega = 0.4$. (3p)

(c) Compare and discuss your results. (1p)

6. Three data series were collected from the website of Statistics Sweden (SCB): The yearly consumption of nuclear power in tera joule (Figure 1), the yearly number of university exams (Figure 2) and the monthly number of employed people, in thousands (Figure 3).

In Figures 4-7, estimated spectral densities of the series of Figures 1-3 are given in 'random' order, together with an estimated spectral density from another series (not shown).

Match figures 1-3 with three of the figures 4-7. Motivate your answer. (5p)

7. Consider the ARCH model

$$y_t = \sigma_t \epsilon_t,$$
$$\sigma_t^2 = 1 + 0.2y_{t-1}^2,$$

where the ϵ_t are i.i.d. $N(0, 1)$.

(a) Show that $E(y_t) = 0$. (2p)

(b) Calculate $E(y_t^6)$. (4p)

Without proof, you may assume that y_t is stationary, $E(y_t^2) = 5/4$, $E(y_t^4) = 225/44$ and $E(\epsilon_t^6) = 15$.

GOOD LUCK!

Appendix: figures

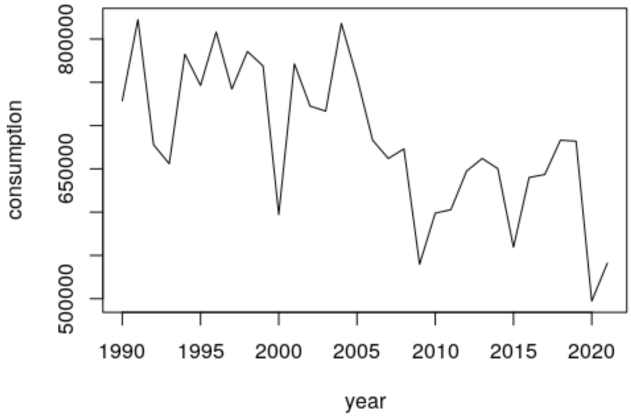

Figure 1: Nuclear power consumption.

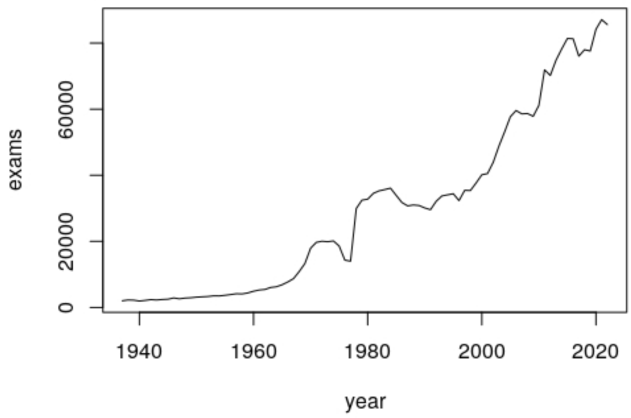

Figure 2: Number of university exams.

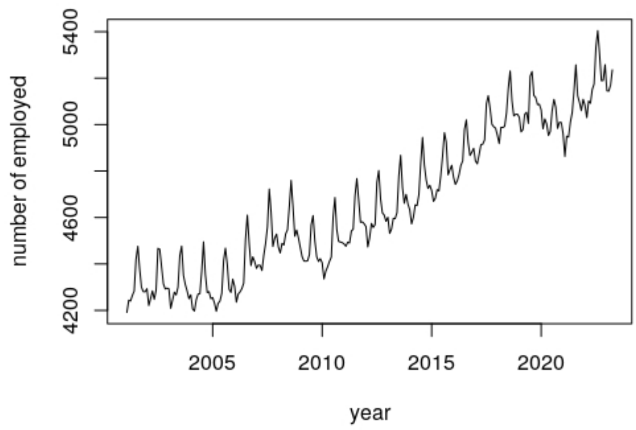

Figure 3: Number of employed people in thousands.

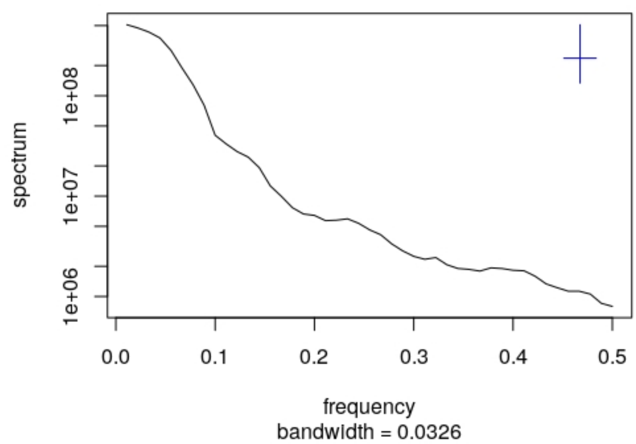

Figure 4: Estimated spectral density, problem 6.

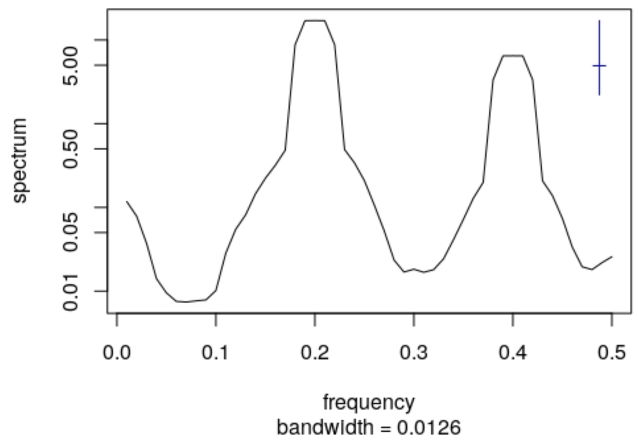

Figure 5: Estimated spectral density, problem 6.

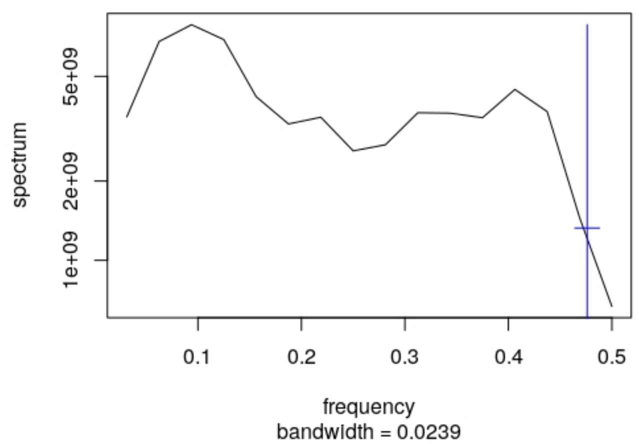

Figure 6: Estimated spectral density, problem 6.

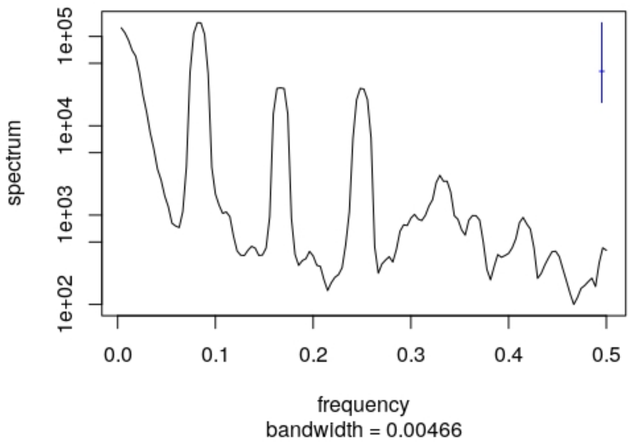

Figure 7: Estimated spectral density, problem 6.