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Exam in Mathematical Statistics
Analysis of Time Series 1MS014
2023–05–30
Solutions

1. Let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 4$ and let
$$x_t = 3 + 0.5w_t^2 + 0.3w_{t-1}^2.$$
Calculate the mean and autocovariance function of x_t and state whether it is weakly stationary. (5p)

Solution: The mean function is given by
$$\mu_t = E(x_t) = 3 + 0.5E(w_t^2) + 0.3E(w_{t-1}^2) = 3 + 0.5 \cdot 4 + 0.3 \cdot 4 = 6.2.$$
As for the autocovariance function, since the covariance between a constant and a random variable is always zero, and because for all t,
$$\text{cov}(w_t^2, w_t^2) = E(w_t^4) - \{E(w_t^2)\}^2 = 3\sigma_w^4 - \sigma_w^4 = 2\sigma_w^4 = 2 \cdot 4^2 = 32,$$
while by independence, $\text{cov}(w_{t+h}^2, w_t^2) = 0$ if $h \neq 0$, we have
$$\begin{aligned}
&\gamma(t+h, t) \\
&= \text{cov}(x_{t+h}, x_t) = \text{cov}(3 + 0.5w_{t+h}^2 + 0.3w_{t+h-1}^2, 3 + 0.5w_t^2 + 0.3w_{t-1}^2) \\
&= 0.5^2\text{cov}(w_{t+h}^2, w_t^2) + 0.5 \cdot 0.3\text{cov}(w_{t+h}^2, w_{t-1}^2) \\
&+ 0.3 \cdot 0.5\text{cov}(w_{t+h-1}^2, w_t^2) + 0.3^2\text{cov}(w_{t+h-1}^2, w_{t-1}^2) \\
&= 0.25 \cdot 32I\{h = 0\} + 0.15 \cdot 32I\{h = -1\} + 0.15 \cdot 32I\{h = 1\} \\
&+ 0.09 \cdot 32I\{h = 0\} \\
&= 10.88I\{h = 0\} + 4.8I\{|h| = 1\} \\
&= \begin{cases} 10.88 & \text{if } h = 0, \\ 4.8 & \text{if } |h| = 1, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$
where $I\{A\} = 1$ if A is fulfilled and 0 otherwise.

Because μ_t is constant and $\gamma(t+h, t)$ is not a function of t, x_t is weakly stationary.

2. For the ARMA(p,q) models below, where $\{w_t\}$ are white noise processes, find p and q and determine whether they are causal and/or invertible. (6p)

   (a) $x_t = 0.8x_{t-1} + w_t$

   Solution: This is an AR(1) model, hence $p = 1$ and $q = 0$. It is invertible, since all AR models are.
   Writing $\theta(B)x_t = w_t$, where $\theta(B) = 1 - 0.8B$, we find that $0 = \theta(z) = 1 - 0.8z$ has the solution $z = 1/0.8 = 1.25$. Since $|1.25| > 1$ (the solution is outside the complex unit circle), the model is causal.

   (b) $x_t = w_t + 0.4w_{t-1} + 0.05w_{t-2}$

   Solution: This is an MA(2) model, i.e. $p = 0$ and $q = 2$. It is causal, since all MA models are.
   To see if it is invertible, we write $x_t = \theta(B)w_t$, where $\theta(B) = 1 + 0.4B + 0.05B^2$. Now, the equation $0 = \theta(z) = 1 + 0.4z + 0.05z^2$ can be written as $z^2 + 8z + 20 = 0$, with solutions

$$z_{1,2} = -4 \pm \sqrt{16 - 20} = -4 \pm 2i,$$

   where $i = \sqrt{-1}$. Hence, $|z_{1,2}|^2 = 4^2 + 2^2 = 20 > 1$, and we find that the model is invertible.

   (c) $x_t = 0.5x_{t-1} + 0.5x_{t-2} + w_t - w_{t-1}$

   Solution: At first, this appears to be an ARMA(2,1) model, $\phi(B)x_t = \theta(B)w_t$ where $\phi(B) = 1 - 0.5B - 0.5B^2$ and $\theta(B) = 1 - B$, but we need to check if $\phi(z) = 0$ and $\theta(z) = 0$ have any common solutions. The latter equation, $1 - z = 0$, has solution $z = 1$, and we find that $\phi(1) = 0$, so in fact, this is the case. It is easy to see that

$$\phi(z) = 1 - 0.5z - 0.5z^2 = (1 - z)(1 + 0.5z),$$

   which means that we can cancel out the factor $1-B$ in the model equation and arrive at $(1 + 0.5B)x_t = w_t$, i.e. $x_t = -0.5x_{t-1} + w_t$.
   This is an AR(1) model, i.e. $p = 1$ and $q = 0$. It is invertible, since all AR models are. Moreover, $0 = \phi(z) = 1 + 0.5z$ has the solution $z = -2$, and $|-2| = 2 > 1$ implying that the model is also causal.

   (d) $x_t = 0.6x_{t-1} + 0.4x_{t-2} + w_t + w_{t-4}$

   Solution: This seems to be an ARMA(2,4) model, i.e. $p = 2$ and $q = 4$, with $\phi(B)x_t = \theta(B)w_t$ where $\phi(B) = 1-0.6B-0.4B^2$ and $\theta(B) = 1+B^4$. The solutions of $0 = \theta(z) = 1 + z^4$ are $\pm(1 \pm i)/\sqrt{2}$, and none of these solve $\phi(z) = 0$, so this is indeed true.
   Alternatively, we can write the model as SARMA$(2,0) \times (0,1)_4$.
   The solutions to $\theta(z) = 0$ given above all have modulus 1, hence they are on the complex unit circle, not outside, and the model is non invertible.
   To check causality, we need to solve $0 = \phi(z) = 1 - 0.6z - 0.4z^2$, i.e.

$$z^2 + \frac{3}{2}z - \frac{5}{2} = 0,$$

which has the solutions

$$z_{1,2} = -\frac{3}{4} \pm \sqrt{\left(\frac{3}{4}\right)^2 + \frac{5}{2}} = \frac{-3 \pm 7}{4},$$

i.e. $z_1 = -5/2$ and $z_2 = 1$. Hence z_2 is on the complex unit circle, and the model is not causal.

3. Let $\{w_t\}$ be a white noise process with variance $\sigma_w^2 = 1$ and define x_t through

$$x_t = 0.5x_{t-4} + w_t + 0.5w_{t-1}.$$

Calculate the autocorrelation function $\rho(h)$ for $h = 1, 2, 3, 4$. (5p)

Solution: At first, we derive $\gamma(0)$ from

$$\begin{aligned}\gamma(0) &= \mathrm{cov}(x_t, x_t) = \mathrm{cov}(0.5x_{t-4} + w_t + 0.5w_{t-1}, 0.5x_{t-4} + w_t + 0.5w_{t-1}) \\ &= 0.5^2\mathrm{cov}(x_{t-4}, x_{t-4}) + \mathrm{cov}(w_t, w_t) + 0.5^2\mathrm{cov}(w_{t-1}, w_{t-1}) \\ &= 0.25\gamma(0) + 1.25,\end{aligned}$$

which yields

$$\gamma(0) = \frac{1.25}{0.75} = \frac{5/4}{3/4} = \frac{5}{3}. \tag{1}$$

In general, we have

$$\begin{aligned}\gamma(h) &= \mathrm{cov}(x_{t+h}, x_t) = \mathrm{cov}(0.5x_{t+h-4} + w_{t+h} + 0.5w_{t+h-1}, x_t) \\ &= 0.5\gamma(h-4) + \mathrm{cov}(w_{t+h}, x_t) + 0.5\mathrm{cov}(w_{t+h-1}, x_t).\end{aligned} \tag{2}$$

With $h = 1$ in (2), using $\gamma(-h) = \gamma(h)$, we get

$$\gamma(1) = 0.5\gamma(3) + 0.5\mathrm{cov}(w_t, x_t),$$

where

$$\mathrm{cov}(w_t, x_t) = \mathrm{cov}(w_t, 0.5x_{t-4} + w_t + 0.5w_{t-1}) = \mathrm{cov}(w_t, w_t) = 1,$$

implying

$$\gamma(1) = 0.5\gamma(3) + 0.5. \tag{3}$$

Now, (2) implies

$$\gamma(3) = 0.5\gamma(1), \tag{4}$$

which we may plug in into (3) to get

$$\gamma(1) = 0.5^2\gamma(1) + 0.5,$$

and so,

$$\gamma(1) = \frac{0.5}{0.75} = \frac{2/4}{3/4} = \frac{2}{3}, \tag{5}$$

and via (1),

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{2/3}{5/3} = \frac{2}{5} = 0.4.$$

Moreover, by (2),

$$\gamma(2) = 0.5\gamma(2),$$

i.e. $\gamma(2) = 0$ and $\rho(2) = 0$. From (4) and (5), we have

$$\gamma(3) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3},$$

so that via (1),

$$\rho(3) = \frac{\gamma(3)}{\gamma(0)} = \frac{1/3}{5/3} = \frac{1}{5} = 0.2.$$

Finally, (2) yields

$$\rho(4) = \frac{\gamma(4)}{\gamma(0)} = \frac{0.5\gamma(0)}{\gamma(0)} = 0.5.$$

4. Consider the process

$$x_t = -0.4x_{t-1} + w_t - 0.7w_{t-1} + 0.1w_{t-2}$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 0.1$. We observe x_t up to time $t = 100$, where the last four observations are $x_{97} = -0.1$, $x_{98} = -0.2$, $x_{99} = -0.2$ and $x_{100} = -0.1$.

(a) Predict the values of x_{101} and x_{102}. Approximations are permitted. (4p)

Solution: We will calculate truncated predictions by using the AR representation $\pi(B)x_t = w_t$. We have

$$(1 - 0.7B + 0.1B^2)w_t = (1 + 0.4B)x_t,$$

which yields

$$\pi(B)(1 - 0.7B + 0.1B^2)w_t = (1 + 0.4B)\pi(B)x_t = (1 + 0.4B)w_t.$$

Hence, with $\pi(z) = 1 + \pi_1 z + \pi_2 z^2 + ...$, we need to solve

$$(1 + \pi_1 z + \pi_2 z^2 + \pi_3 z^3 + ...)(1 - 0.7z + 0.1z^2) = 1 + 0.4z,$$

i.e.

$$1+(\pi_1-0.7)z+(\pi_2-0.7\pi_1+0.1)z^2+(\pi_3-0.7\pi_2+0.1\pi_1)z^3+... = 1+0.4z,$$

which yields

$$\begin{aligned}
\pi_1 &= 0.7 + 0.4 = 1.1,\\
\pi_2 &= 0.7\pi_1 - 0.1 = 0.67,\\
\pi_3 &= 0.7\pi_2 - 0.1\pi_1 = 0.359,\\
\pi_4 &= 0.7\pi_3 - 0.1\pi_2 = 0.1843,\\
\pi_5 &= 0.7\pi_4 - 0.1\pi_3 = 0.09311.
\end{aligned}$$

The truncated predictions become

$$\begin{aligned}
\tilde{x}_{101} &= -\pi_1 x_{100} - \pi_2 x_{99} - ...\\
&\approx (-1.1)\cdot(-0.1) + (-0.67)\cdot(-0.2) + (-0.359)\cdot(-0.2)\\
&\quad + (-0.1843)\cdot(-0.1) = 0.33423
\end{aligned}$$

and

$$\begin{aligned}
\tilde{x}_{102} &= -\pi_1\tilde{x}_{101} - \pi_2 x_{100} - \pi_3 x_{99} - ...\\
&\approx (-1.1)\cdot 0.33423 + (-0.67)\cdot(-0.1) + (-0.359)\cdot(-0.2)\\
&\quad + (-0.1843)\cdot(-0.2) + (-0.09311)\cdot(-0.1)\\
&= -0.182682.
\end{aligned}$$

(b) Calculate 95% prediction intervals for x_{101} and x_{102}. (3p)

Solution: The mean square predicion error m steps ahead is given by $\sigma_w^2 \sum_{j=1}^{m-1} \psi_j^2$, where the ψ_j are the coefficients in the MA representation, with $\psi_0 = 1$. We only need to find ψ_1. To this end, $x_t = \psi(B) w_t$ implies

$$\begin{aligned}\psi(B)(1 + 0.4B)x_t &= (1 - 0.7B + 0.1B^2)\psi(B)w_t \\ &= (1 - 0.7B + 0.1B^2)x_t,\end{aligned}$$

so that with $\psi(z) = 1 + \psi_1 z + ...$, we have

$$(1 + \psi_1 z + ...)(1 + 0.4z) = 1 - 0.7z + 0.1z^2,$$

implying $\psi_1 + 0.4 = -0.7$, i.e. $\psi_1 = -1.1$.
With $\sigma_w^2 = 0.1$, this gives the 95% prediction interval for x_{101} as

$$0.33423 \pm 1.96\sqrt{0.1} = 0.33423 \pm 0.61981 = (-0.286,\ 0.954),$$

and for x_{102}, we find the corresponding interval

$$\begin{aligned}-0.182682 \pm 1.96\sqrt{0.1(1 + 1.1^2)} &= -0.182682 \pm 0.92141 \\ &= (-1.104,\ 0.739).\end{aligned}$$

5. A time series $\{x_t\}$ follows the model

$$x_t = 0.2x_{t-1} + w_t,$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 1$. This series is used as input for constructing

$$y_t = 0.8y_{t-1} + x_t,$$

for $t = 0, \pm 1, \pm 2, ...$

(a) Calculate the spectral densitiy of x_t at the frequencies $\omega = 0.1$ and $\omega = 0.4$. (2p)

Solution: We have $\phi(B)x_t = w_t$ with $\phi(B) = 1 - 0.2B$. We want to use the formula

$$f_x(\omega) = \frac{\sigma_w^2}{|\phi(e^{-2\pi i\omega})|^2}.$$

Here, $\sigma_w^2 = 1$ and

$$\begin{aligned}|\phi(e^{-2\pi i\omega})|^2 &= |1 - 0.2e^{-2\pi i\omega}|^2 = (1 - 0.2e^{-2\pi i\omega})(1 - 0.2e^{2\pi i\omega})\\ &= 1 + 0.2^2 - 0.4\frac{e^{2\pi i\omega} + e^{-2\pi i\omega}}{2} = 1.04 - 0.4\cos(2\pi\omega).\end{aligned}$$

Hence,

$$f_x(\omega) = \frac{1}{1.04 - 0.4\cos(2\pi\omega)},$$

and insertion gives $f_x(0.1) \approx 1.396$ and $f_x(0.4) \approx 0.733$.

(b) Calculate the spectral densitiy of y_t at the frequencies $\omega = 0.1$ and $\omega = 0.4$. (3p)

Solution: We will find the filter $y_t = \sum_j a_j x_{t-j}$ and then use the frequency response function. To find the filter, we use recursion to get

$$\begin{aligned}y_t &= 0.8y_{t-1} + x_t = 0.8(0.8y_{t-2} + x_{t-1}) + x_t = 0.8^2 y_{t-2} + 0.8x_{t-1} + x_t\\ &= ... = \sum_{j=0}^{\infty} 0.8^j x_{t-j},\end{aligned}$$

i.e. $a_j = 0.8^j$ for $j = 0, 1, 2, ...$ and 0 for $j < 0$. Hence, we have the frequency response function

$$A(\omega) = \sum_{j=0}^{\infty} a_j e^{-2\pi i\omega j} = \sum_{j=0}^{\infty} (0.8e^{-2\pi i\omega})^j = \frac{1}{1 - 0.8e^{-2\pi i\omega}}.$$

This means that

$$\begin{aligned}|A(\omega)|^2 = A(\omega)\overline{A(\omega)} &= \frac{1}{(1 - 0.8e^{-2\pi i\omega})(1 - 0.8e^{2\pi i\omega})}\\ &= \frac{1}{1.64 - 1.6\cos(2\pi\omega)}.\end{aligned}$$

Hence, the formula

$$f_y(\omega) = |A(\omega)|^2 f_x(\omega)$$

yields $f_y(0.1) \approx 4.039$ and $f_y(0.4) \approx 0.250$.

(c) Compare and discuss your results. (1p)

Solution: The filter in (b) is recursive, with relatively high weights, meaning that is 'smoothes out' the series relatively much. Hence, it is a low-pass filter, decreasing the weights for the high frequencies relative to the low ones. This is what we see from our numbers, where the spectral density at 0.1 increases after filtering, while it decreases at 0.4.

6. Three data series were collected from the website of Statistics Sweden (SCB): The yearly consumption of nuclear power in tera joule (Figure 1), the yearly number of university exams (Figure 2) and the monthly number of employed people, in thousands (Figure 3).

In Figures 4-7, estimated spectral densities of the series of Figures 1-3 are given in 'random' order, together with an estimated spectral density from another series (not shown).

Match figures 1-3 with three of the figures 4-7. Motivate your answer. (5p)

Solution: The series in figure 1 shows no particular pattern, while the series in figure 2 has an obvious upward trend, and the series in figure 3 has both a trend and a season of 12 (monthly data).

The estimated spectral densiy of figure 4 has high weight on low frequencies but no seasonal peaks, thus it should correspond to the trending series in figure 2.

In figure 5, we find a low weight on low frequencies and peaks at 0.2 and 0.4, which should correspond to a series with period length 5 and no particular trend. We can not clearly see this pattern in any of figures 1-3.

We can move to figure 7, where we have high values for low frequencies, indicating trend, and high values on peaks that are multiples of about 0.08, so this seems to correspond to the monthly series with a trend of figure 3.

Finally, figure 6 does not show much of a structure, except perhaps a small peak around 0.1. This should correspond to a period length of about 10. The series in figure 1 has a little bit of that, so it seems to be the best fit. (At least, the series in figure 1 doesn't show clear signs of a period length of 5, which would correspond to figure 5.)

7. Consider the ARCH model

$$
\begin{aligned}
y_t &= \sigma_t \epsilon_t, \\
\sigma_t^2 &= 1 + 0.2 y_{t-1}^2,
\end{aligned}
$$

where the ϵ_t are i.i.d. $N(0,1)$.

(a) Show that $E(y_t) = 0$. (2p)

Solution: Let $Y_s = \{y_s, y_{s-1}, ...\}$. i.e. all information gathered up to time s. By the law of iterated expectations, we have

$$
\begin{aligned}
E(y_t) &= E\{E(y_t|Y_{t-1})\} = E\{E(\sigma_t \epsilon_t|Y_{t-1})\} = E\{\sigma_t E(\epsilon_t|Y_{t-1})\} \\
&= E\{\sigma_t E(\epsilon_t)\} = 0.
\end{aligned}
$$

Here, the third equality follows because σ_t is a function of Y_{t-1}, the fourth equality follows because ϵ_t is independent of Y_{t-1}, and the last equality follows since $E(\epsilon_t) = 0$.

(b) Calculate $E(y_t^6)$. (4p)

Without proof, you may assume that y_t is stationary,
$E(y_t^2) = 5/4$, $E(y_t^4) = 225/44$ and $E(\epsilon_t^6) = 15$.

Solution: We will use

$$
E(y_t^6) = E\{E(y_t^6|Y_{t-1})\},
$$

where, similar to (a),

$$
\begin{aligned}
E(y_t^6|Y_{t-1}) &= E(\sigma_t^6 \epsilon_t^6|Y_{t-1}) = \sigma_t^6 E(\epsilon_t^6|Y_{t-1}) = \sigma_t^6 E(\epsilon_t^6) = 15\sigma_t^6 \\
&= 15\left(1 + \frac{1}{5} y_{t-1}^2\right)^3 = 15 + 9y_{t-1}^2 + \frac{9}{5} y_{t-1}^4 + \frac{3}{25} y_{t-1}^6.
\end{aligned}
$$

Hence, using stationarity and the other help information,

$$
E(y_t^6) = 15 + 9 \cdot \frac{5}{4} + \frac{9}{5} \cdot \frac{225}{44} + \frac{3}{25} E(y_t^6) = 15 + \frac{45}{4} + \frac{9 \cdot 45}{44} + \frac{3}{25} E(y_t^6),
$$

and solving for $E(y_t^6)$ yields

$$
E(y_t^6) = \frac{25}{22}\left(15 + \frac{45}{4} + \frac{9 \cdot 45}{44}\right) \approx 40.3.
$$

Appendix: figures

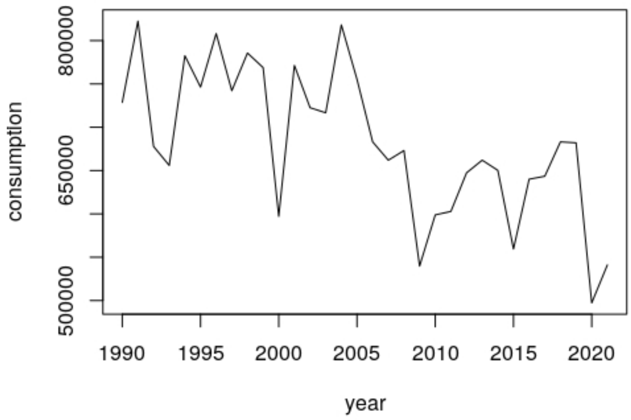

Figure 1: Nuclear power consumption.

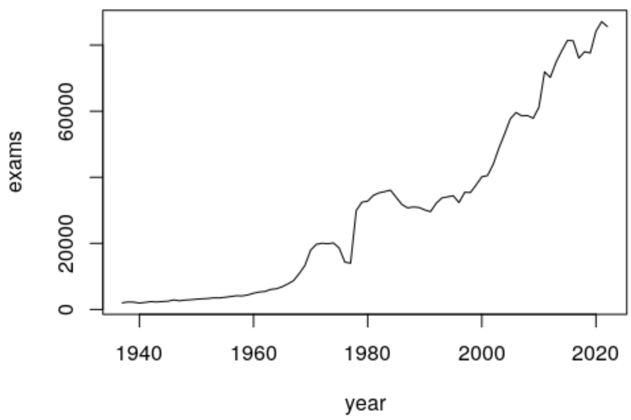

Figure 2: Number of university exams.

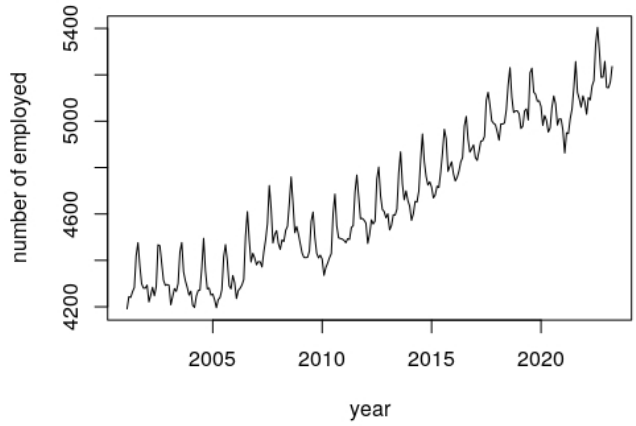

Figure 3: Number of employed people in thousands.

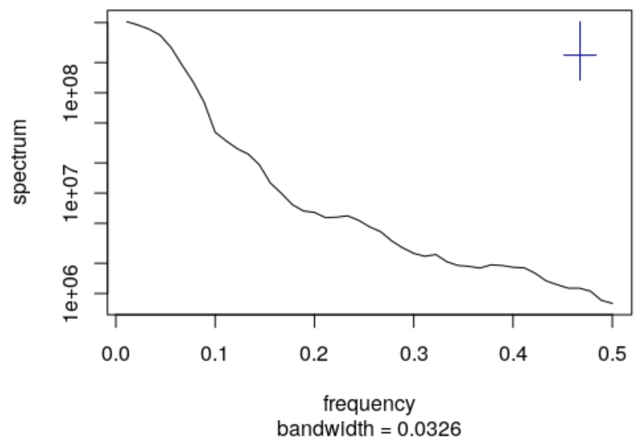

Figure 4: Estimated spectral density, problem 6.

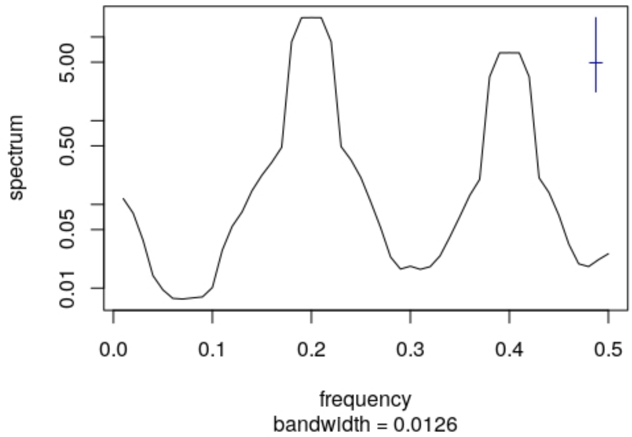

Figure 5: Estimated spectral density, problem 6.

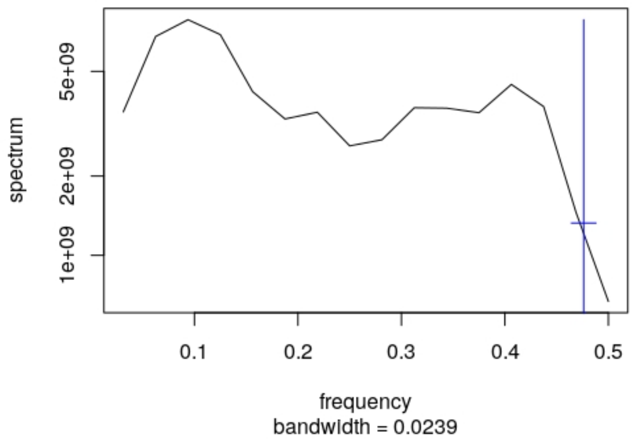

Figure 6: Estimated spectral density, problem 6.

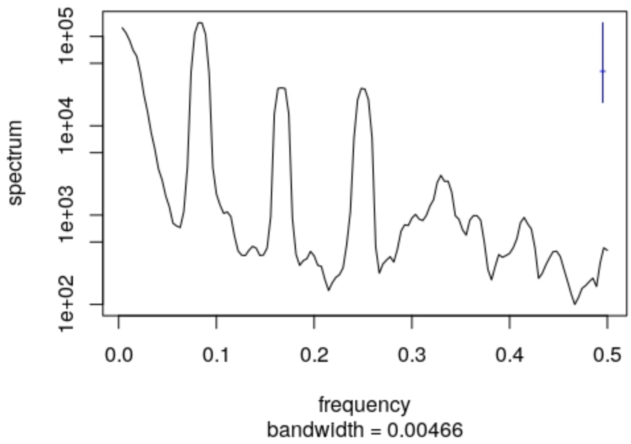

Figure 7: Estimated spectral density, problem 6.