UPPSALA UNIVERSITY
Department of Mathematics
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Exam in Mathematical Statistics
Analysis of Time Series 1MS014
2023–08–17

Time: 8.00-13.00. Limits for the credits 3, 4, 5 are 18, 25 and 32 points, respectively. The solutions should be well motivated.

Permitted aids: The course book or copies thereof. Hand-written sheet of formulae. Pocket calculator. Dictionary. *No electronic device with internet connection.*

1. Let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 2$ and let
$$x_t = 1 + 0.4w_t^2 + 0.1w_{t-1}.$$
Calculate the mean and autocovariance function of x_t and state whether it is weakly stationary. (5p)

2. For the ARMA(p, q) models below, where $\{w_t\}$ are white noise processes, find p and q and determine whether they are causal and/or invertible. (6p)
   (a) $x_t = w_t + 0.5w_{t-1}$
   (b) $x_t = w_t - 1.2w_{t-1} + 0.2w_{t-2}$
   (c) $x_t = 0.5x_{t-1} + w_t - 0.5w_{t-1}$
   (d) $x_t = -0.36x_{t-2} + w_t + 0.4w_{t-1}$

3. Let $\{w_t\}$ be a white noise process with variance $\sigma_w^2 = 1$ and define x_t through
$$x_t = 0.5x_{t-1} + w_t - 0.5w_{t-2}.$$
Calculate the autocorrelation function $\rho(h)$ for $h = 1, 2, 3, 4$. (5p)

4. Consider the process
$$x_t = 0.25x_{t-2} + w_t + 0.4w_{t-1}$$
where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 0.09$. We observe x_t up to time $t = 200$, where the last four observations are $x_{197} = 0.3$, $x_{198} = 0.4$, $x_{199} = 0.5$ and $x_{200} = 0.6$.
   (a) Predict the values of x_{201} and x_{202}. Approximations are permitted. (4p)
   (b) Calculate 95% prediction intervals for x_{201} and x_{202}. (3p)

5. Consider the time series model

$$x_t = 0.2x_{t-1} + 0.4x_{t-4} - 0.08x_{t-5} + w_t,$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 1$.

(a) Write it as a seasonal model with period 4. (2p)

(b) Calculate the spectral density at the frequency $\omega = 0.25$. (2p)

(c) Calculate the spectral density of $y_t = \frac{1}{4}(x_t + x_{t-1} + x_{t-2} + x_{t-3})$ at the frequency $\omega = 0.25$. (2p)

(d) Compare and discuss your results in (b) and (c). (1p)

6. Four time series of length 200 were generated. Their estimated ACF and PACF are given in figures 1-4 below. Figures 5-8, given in a "random" order, in turn depict their estimated spectral densities (spans=8). Each one of figures 1-4 corresponds to one and only one of figures 5-8.

Match figures 1-4 with figures 5-8. Motivate your answer. (5p)

7. Consider the model

$$x_t = \begin{cases} \alpha^{(1)} + \phi^{(1)}x_{t-1} + w_t^{(1)}, & \text{if } x_{t-1} < 0, \\ \alpha^{(2)} + \phi^{(2)}x_{t-1} + w_t^{(2)}, & \text{if } x_{t-1} \geq 0, \end{cases}$$

where $\{w_t^{(1)}\}$ and $\{w_t^{(2)}\}$ are independent white noise processes.

Describe how to reformulate this model in such a way that the parameters $\alpha^{(1)}$, $\alpha^{(2)}$, $\phi^{(1)}$ and $\phi^{(2)}$ may be estimated by standard methods (like least squares).

(5p)

GOOD LUCK!

Appendix: figures

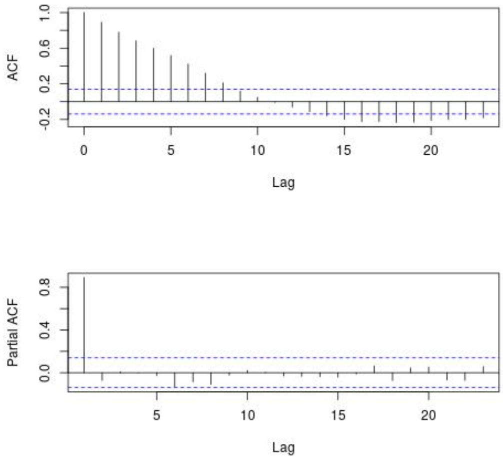

Figure 1: ACF and PACF, problem 6.

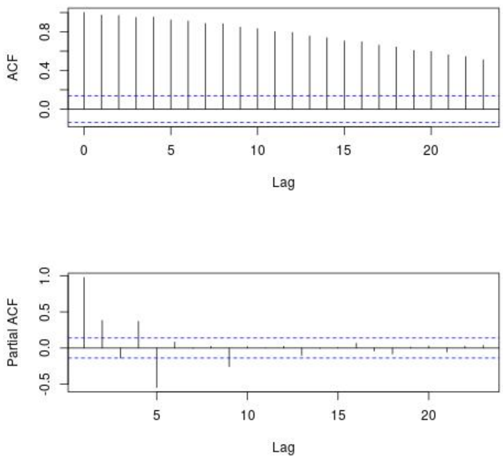

Figure 2: ACF and PACF, problem 6.

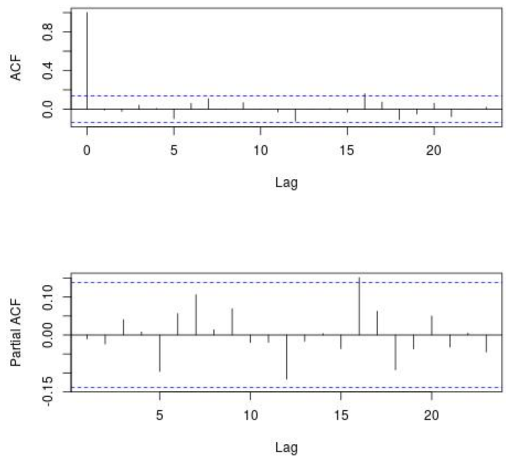

Figure 3: ACF and PACF, problem 6.

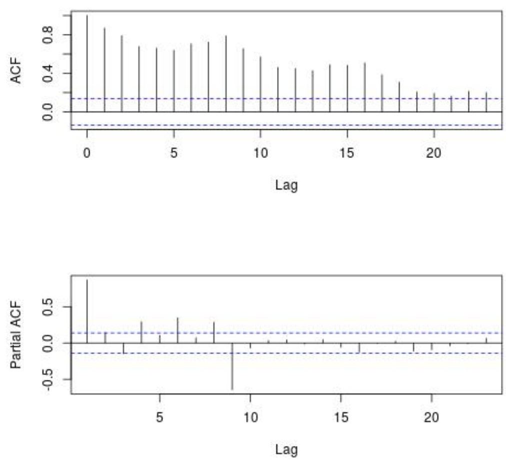

Figure 4: ACF and PACF, problem 6.

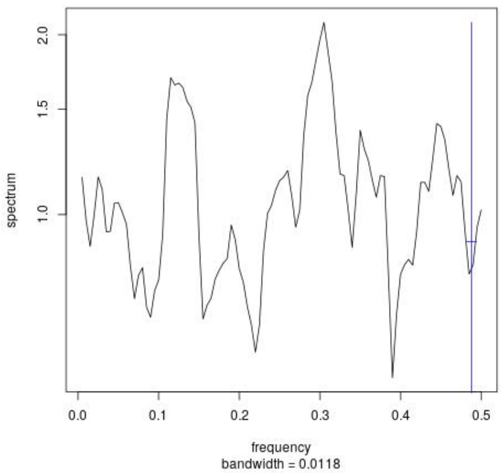

Figure 5: Estimated spectral density, problem 6.

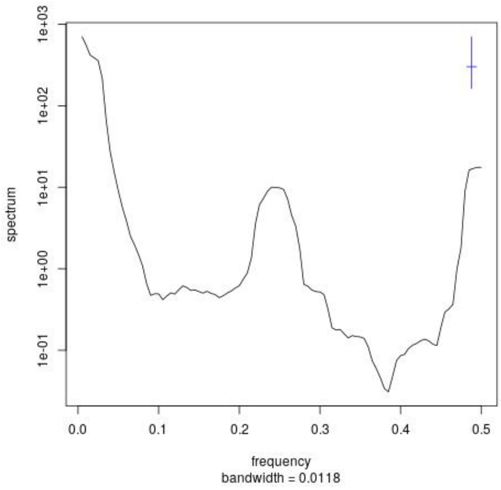

Figure 6: Estimated spectral density, problem 6.

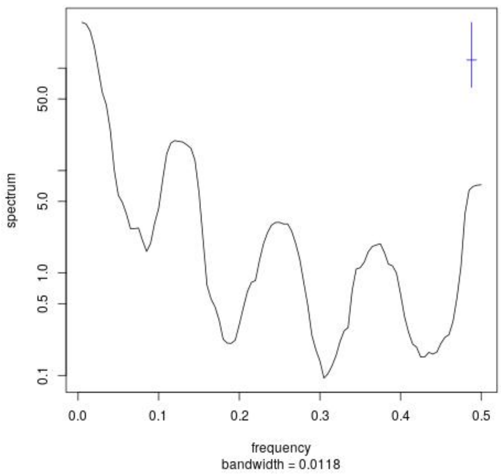

Figure 7: Estimated spectral density, problem 6.

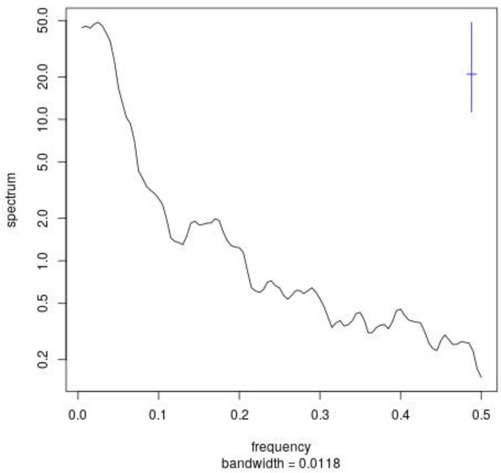

Figure 8: Estimated spectral density, problem 6.