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Exam in Mathematical Statistics
Analysis of Time Series 1MS014
2023–08–17
Solutions

1. Let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 2$ and let
$$x_t = 1 + 0.4w_t^2 + 0.1w_{t-1}.$$
Calculate the mean and autocovariance function of x_t and state whether it is weakly stationary. (5p)

Solution: The mean function is given by
$$\mu_t = E(x_t) = 1 + 0.4E(w_t^2) + 0.1E(w_{t-1}) = 1 + 0.4 \cdot 2 + 0.1 \cdot 0 = 1.8.$$
Moreover, the autocovariance function may be written as
$$\begin{aligned}\gamma(t+h, t) &= \text{cov}(x_{t+h}, x_t) \\ &= \text{cov}(1 + 0.4w_{t+h}^2 + 0.1w_{t+h-1}, 1 + 0.4w_t^2 + 0.1w_{t-1}) \\ &= 0.4^2\text{cov}(w_{t+h}^2, w_t^2) + 0.4 \cdot 0.1\text{cov}(w_{t+h}^2, w_{t-1}) \\ &+ 0.1 \cdot 0.4\text{cov}(w_{t+h-1}, w_t^2) + 0.1^2\text{cov}(w_{t+h-1}^2, w_{t-1}^2). \qquad (1)\end{aligned}$$
Here, we have that for $h \neq 0$, $\text{cov}(w_{t+h}^2, w_t^2) = 0$ because of independence, while for $h = 0$, it equals
$$\text{var}(w_t^2) = E(w_t^4) - \{E(w_t^2)\}^2 = 3 \cdot 2^2 - 2^2 = 8.$$
For the same reason, for $h \neq -1$, $\text{cov}(w_{t+h}^2, w_{t-1}) = 0$, while for $h = -1$ it is
$$\text{cov}(w_{t-1}^2, w_{t-1}) = E(w_{t-1}^3) - E(w_{t-1}^2)E(w_{t-1}) = 0,$$
since all odd moments of w_{t-1} are zero. Hence, from (1),
$$\gamma(t+h, t) = \begin{cases} (0.4^2 + 0.1^2) \cdot 8 = 1.36 & \text{if } h = 0, \\ 0 & \text{otherwise.} \end{cases}$$
Since the variance $\gamma(t, t)$ is finite and none of μ_t or $\gamma(t+h, t)$ is a function of t, we conclude that x_t is weakly stationary.

2. For the ARMA(p, q) models below, where $\{w_t\}$ are white noise processes, find p and q and determine whether they are causal and/or invertible. (6p)

   (a) $x_t = w_t + 0.5w_{t-1}$

   Solution: This is an MA(1) model, i.e. $p = 0$, $q = 1$. It is casual, since all MA models are. We write $x_t = \theta(B)w_t$, where $\theta(B) = 1+0.5B$. The root of the MA polynomial is found from $1-0.5z = 0$, which gives $z = 2$. This root is outside the complex unit circle. Hence, the model is invertible.

   (b) $x_t = w_t - 1.2w_{t-1} + 0.2w_{t-2}$

   Solution: This is an MA(2) model, i.e. $p = 0$, $q = 2$. It is casual, since all MA models are. We write $x_t = \theta(B)w_t$, where $\theta(B) = 1 - 1.2B + 0.2B^2$. The roots of the MA polynomial are found from $1 - 1.2z + 0.2z^2 = 0$, or equivalently,

$$z^2 - 6z + 5 = 0,$$

   with solutions

$$z_{1,2} = 3 \pm \sqrt{3^2 - 5} = 3 \pm 2,$$

   hence $z_1 = 1$, $z_2 = 5$. Because $z_1 = 1$ is on the complex unit circle, the model is *not* invertible.

   (c) $x_t = 0.5x_{t-1} + w_t - 0.5w_{t-1}$

   Solution: This appears to be an ARMA(1,1) model, but writing the model with operators, it is

$$(1 - 0.5B)x_t = (1 - 0.5B)w_t,$$

   which is equivalent with $x_t = w_t$. Hence, we have white noise, i.e. $p = 0$, $q = 0$. White noise is both causal and invertible.

   (d) $x_t = -0.36x_{t-2} + w_t + 0.4w_{t-1}$

   Solution: This is an ARMA(2,1) model, i.e. $p = 2$, $q = 1$. On operator form, it is $\phi(B)x_t = \theta(B)w_t$ with $\phi(B) = 1+0.36B^2$ and $\theta(B) = 1+0.4B$. The AR polynomial roots are found through $1 + 0.36z^2 = 0$, which gives

$$z_{1,2} = \pm\frac{i}{0.6},$$

   where $i^2 = -1$. Hence, $|z_{1,2}|^2 = 1/0.36 > 1$, which means that the roots are both outside the complex unit circle. Consequently, the model is causal.

   The MA ploynomial root is found from $1+0.4z = 0$, i.e. $z = -2.5$, which is outside the complex unit circle, proving that the model is invertible as well.

3. Let $\{w_t\}$ be a white noise process with variance $\sigma_w^2 = 1$ and define x_t through

$$x_t = 0.5x_{t-1} + w_t - 0.5w_{t-2}.$$

Calculate the autocorrelation function $\rho(h)$ for $h = 1, 2, 3, 4$. (5p)

Solution: We have the autocovariance function

$$\begin{aligned}
\gamma(h) &= \mathrm{cov}(x_t, x_{t-h}) = \mathrm{cov}(0.5x_{t-1} + w_t - 0.5w_{t-2}, x_{t-h}) \\
&= 0.5\mathrm{cov}(x_{t-1}, x_{t-h}) + \mathrm{cov}(w_t, x_{t-h}) - 0.5\mathrm{cov}(w_{t-2}, x_{t-h}) \\
&= 0.5\gamma(h-1) + \mathrm{cov}(w_t, x_{t-h}) - 0.5\mathrm{cov}(w_{t-2}, x_{t-h}),
\end{aligned}$$

from which we deduce, using $\gamma(-h) = \gamma(h)$ and the fact that w_t is independent of x_{t-h} for $h > 0$,

$$\begin{aligned}
\gamma(0) &= 0.5\gamma(1) + \mathrm{cov}(w_t, x_t) - 0.5\mathrm{cov}(w_{t-2}, x_t), && (2) \\
\gamma(1) &= 0.5\gamma(0) - 0.5\mathrm{cov}(w_{t-2}, x_{t-1}), && (3) \\
\gamma(2) &= 0.5\gamma(1) - 0.5\mathrm{cov}(w_{t-2}, x_{t-2}), && (4)
\end{aligned}$$

and we also have

$$\gamma(h) = 0.5\gamma(h-1), \;\; h \geq 3. \qquad (5)$$

Similarly, we obtain

$$\begin{aligned}
\mathrm{cov}(x_t, w_t) &= \mathrm{cov}(0.5x_{t-1} + w_t - 0.5w_{t-2}, w_t) = \mathrm{cov}(w_t, w_t) = 1, \\
\mathrm{cov}(x_t, w_{t-1}) &= 0.5\mathrm{cov}(x_{t-1}, w_{t-1}) = 0.5\mathrm{cov}(x_t, w_t) = 0.5, \\
\mathrm{cov}(x_t, w_{t-2}) &= 0.5\mathrm{cov}(x_{t-1}, w_{t-2}) - 0.5\mathrm{cov}(w_{t-2}, w_{t-2}) \\
&= 0.5 \cdot 0.5 - 0.5 = -0.25.
\end{aligned}$$

Now, (2)-(4) imply

$$\begin{aligned}
\gamma(0) &= 0.5\gamma(1) + 1 + 0.5 \cdot 0.25 = 0.5\gamma(1) + 1.125, && (6) \\
\gamma(1) &= 0.5\gamma(0) - 0.5 \cdot 0.5 = 0.5\gamma(0) - 0.25, && (7) \\
\gamma(2) &= 0.5\gamma(1) - 0.5. && (8)
\end{aligned}$$

Putting (7) in (6), we get

$$\gamma(0) = 0.5\{0.5\gamma(0) - 0.25\} + 1.125 = 0.25\gamma(0) + 1,$$

which yields

$$\gamma(0) = \frac{1}{0.75} = \frac{4}{3}.$$

Then, inserting into (7),

$$\gamma(1) = \frac{1}{2} \cdot \frac{4}{3} - \frac{1}{4} = \frac{5}{12},$$

implying

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{5/12}{4/3} = \frac{5}{16} = 0.3125.$$

Similarly,

$$\gamma(2) = \frac{1}{2} \cdot \frac{5}{12} - \frac{1}{2} = -\frac{7}{24},$$

which yields

$$\rho(2) = \frac{\gamma(2)}{\gamma(0)} = -\frac{7/24}{4/3} = -\frac{7}{32} = -0.21875,$$

and (5) divided by $\gamma(0)$ implies

$$\rho(3) = \frac{1}{2}\rho(2) = -\frac{7}{64} = -0.109375$$

and

$$\rho(4) = \frac{1}{2}\rho(3) = -\frac{7}{128} = -0.0546875.$$

4. Consider the process

$$x_t = 0.25x_{t-2} + w_t + 0.4w_{t-1},$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 0.09$. We observe x_t up to time $t = 200$, where the last four observations are $x_{197} = 0.3$, $x_{198} = 0.4$, $x_{199} = 0.5$ and $x_{200} = 0.6$.

(a) Predict the values of x_{201} and x_{202}. Approximations are permitted. (4p)

Solution: We will calculate truncated predictions by using the AR representation $\pi(B)x_t = w_t$. We have

$$(1 + 0.4B)w_t = (1 - 0.25B^2)x_t,$$

which yields

$$\pi(B)(1 + 0.4B)w_t = (1 - 0.25B^2)\pi(B)x_t = (1 - 0.25B^2)w_t.$$

Hence, with $\pi(z) = 1 + \pi_1 z + \pi_2 z^2 + ...$, we need to solve

$$(1 + \pi_1 z + \pi_2 z^2 + ...)(1 + 0.4z) = 1 - 0.25z^2,$$

i.e.

$$1 + (\pi_1 + 0.4)z + (\pi_2 + 0.4\pi_1)z^2 + ... = 1 - 0.25z^2,$$

which yields

$$\begin{aligned}
\pi_1 &= -0.4, \\
\pi_2 &= -0.4\pi_1 - 0.25 = -0.09, \\
\pi_3 &= -0.4\pi_2 = 0.036, \\
\pi_4 &= -0.4\pi_3 = -0.0144, \\
\pi_5 &= -0.4\pi_4 = 0.00576.
\end{aligned}$$

The truncated predictions become

$$\begin{aligned}
\tilde{x}_{201} &= -\pi_1 x_{200} - \pi_2 x_{199} - ... \\
&\approx 0.4 \cdot 0.6 + 0.09 \cdot 0.5 - 0.036 \cdot 0.4 + 0.0144 \cdot 0.3 \\
&= 0.27492
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{x}_{202} \\
&= -\pi_1 \tilde{x}_{201} - \pi_2 x_{200} - \pi_3 x_{199} - ... \\
&\approx 0.4 \cdot 0.27492 + 0.09 \cdot 0.6 - 0.036 \cdot 0.5 + 0.0144 \cdot 0.4 - 0.00576 \cdot 0.3 \\
&= 0.15.
\end{aligned}$$

(b) Calculate 95% prediction intervals for x_{201} and x_{202}. (3p)

Solution: The mean square predicion error m steps ahead is given by $\sigma_w^2 \sum_{j=1}^{m-1} \psi_j^2$, where the ψ_j are the coefficients in the MA representation, with $\psi_0 = 1$. We only need to find ψ_1. To this end, $x_t = \psi(B) w_t$ implies

$$\begin{aligned}\psi(B)(1 - 0.25B^2)x_t &= (1 + 0.4B)\psi(B)w_t \\ &= (1 + 0.4B)x_t,\end{aligned}$$

so that with $\psi(z) = 1 + \psi_1 z + ...$, we have

$$(1 + \psi_1 z + ...)(1 - 0.25z^2) = 1 + 0.4z,$$

implying $\psi_1 = 0.4$.
With $\sigma_w^2 = 0.09$, this gives the 95% prediction interval for x_{201} as

$$0.27492 \pm 1.96\sqrt{0.09} = 0.27492 \pm 0.588 = (-0.313,\ 0.863),$$

and for x_{202}, we find the corresponding interval

$$0.15 \pm 1.96\sqrt{0.09(1 + 0.4^2)} = 0.15 \pm 0.633 = (-0.483,\ 0.783).$$

5. Consider the time series model

$$x_t = 0.2x_{t-1} + 0.4x_{t-4} - 0.08x_{t-5} + w_t,$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 1$.

(a) Write it as a seasonal model with period 4. (2p)

Solution: We may write the model as $\phi(B)x_t = w_t$ where

$$\phi(B) = 1 - 0.2B - 0.4B^4 + 0.08B^5.$$

It is quite clear that $\phi(B)$ is on the form

$$(1 - aB)(1 - bB^4) = 1 - aB - bB^4 + abB^5,$$

from which we see that we must have $a = 0.2$ and $b = 0.4$. Note that then, $ab = 0.08$. Hence, we may write the seasonal model as

$$\phi(B)x_t = (1 - 0.2B)(1 - 0.4B^4)x_t = w_t.$$

(b) Calculate the spectral density at the frequency $\omega = 0.25$. (2p)

Solution: We use the general formula

$$f(\omega) = \sigma_w^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2},$$

where in our case, we have $\theta(z) = 1$ and $\phi(z) = (1 - 0.2z)(1 - 0.4z^4)$, so that

$$\begin{aligned}
&|\phi(e^{-2\pi i\omega})|^2 \\
&= \phi(e^{-2\pi i\omega})\phi(e^{2\pi i\omega}) \\
&= (1 - 0.2e^{-2\pi i\omega})(1 - 0.2e^{2\pi i\omega})(1 - 0.4e^{-8\pi i\omega})(1 - 0.4e^{8\pi i\omega}) \\
&= (1 - 0.2e^{2\pi i\omega} - 0.2e^{-2\pi i\omega} + 0.04)(1 - 0.4e^{8\pi i\omega} - 0.4e^{-8\pi i\omega} + 0.16) \\
&= \{1.04 - 0.4\cos(2\pi\omega)\}\{1.16 - 0.8\cos(8\pi\omega)\}.
\end{aligned}$$

Hence, with $\sigma_w^2 = 1$, we have

$$f(\omega) = \frac{1}{\{1.04 - 0.4\cos(2\pi\omega)\}\{1.16 - 0.8\cos(8\pi\omega)\}},$$

which yields

$$f(0.25) = \frac{1}{1.04 \cdot (1.16 - 0.8)} = \frac{1}{0.3744} \approx 2.67.$$

(c) Calculate the spectral density of $y_t = \frac{1}{4}(x_t + x_{t-1} + x_{t-2} + x_{t-3})$ at the frequency $\omega = 0.25$. (2p)

Solution: We have the filter $a_0 = a_1 = a_2 = a_3 = 1/4$ and $a_j = 0$ otherwise, which yields the frequency response function

$$A_{yx}(\omega) = \sum_j a_j e^{-2\pi i\omega j} = \frac{1}{4}(1 + e^{-2\pi i\omega} + e^{-4\pi i\omega} + e^{-6\pi i\omega}),$$

implying in the same manner as above that

$$\begin{aligned}
&|A_{yx}(\omega)|^2 \\
&= \frac{1}{16}(1 + e^{-2\pi i\omega} + e^{-4\pi i\omega} + e^{-6\pi i\omega})(1 + e^{2\pi i\omega} + e^{4\pi i\omega} + e^{6\pi i\omega}) \\
&= \frac{1}{16}\left\{4 + 6\cos(2\pi\omega) + 4\cos(4\pi\omega) + 2\cos(2\pi\omega)\right\},
\end{aligned}$$

and so,

$$|A_{yx}(0.25)|^2 = \frac{1}{16}\left\{4 + 6 \cdot 0 + 4 \cdot (-1) + 2 \cdot 0\right\} = 0.$$

This means that the spectral density of y at $\omega = 0.25$ is

$$f_{yy}(0.25) = |A_{yx}(0.25)|^2 f_{xx}(0.25) = 0.$$

(d) Compare and discuss your results in (b) and (c). (1p)

Solution: It is a seasonal model with period 4, so the spectral density should peak at $\omega = 1/4 = 0.25$. Hence, we get a large value in (b). In (c), we use a seasonal filter, which kills the seasonal frequency and hence, the spectral density at $\omega = 1/4 = 0.25$ is zero.

6. Four time series of length 200 were generated. Their estimated ACF and PACF are given in figures 1-4 below. Figures 5-8, given in a "random" order, in turn depict their estimated spectral densities (spans=8). Each one of figures 1-4 corresponds to one and only one of figures 5-8.

   Match figures 1-4 with figures 5-8. Motivate your answer. (5p)

   Solution: In figure 1, the ACF decays slowly while the PACF has a peak at lag 1 and is close to zero otherwise. So this is a series with a strong AR(1) structure, i.e. with high weights on low frequencies and no seasonal behaviour. This is matched by the spectral density in figure 8.

   Figure 2 has an ACF which is really slowly decaying, indicating a trend or something close to it. The PACF peaks at lag 5, but it is also significant at lags 2, 4 and 9. It seems that the series could have a period of 4 or 5. Together with the trend, this should give a spectral density that is high for low frequencies as well as for frequencies close to 1/4 or 1/5. The best match to this is figure 6.

   In figure 3, in fact none of the ACF or PACF values are significant, except for at lag length 16, which might be an effect of mass significance. So maybe this is white noise or something close to it. Hence, the estimated spectral density should be close to constant and have no specific structure. This is what we have in figure 5 (compare to the confidence interval length shown by the vertical line to the right in the figure).

   It remains to match figures 4 and 7. This is also reasonable, since in figure 4, both the ACF and the PACF peak at lag 8, corresponding to peaks at multiples of $1/8 = 0.125$ in the spectral density of figure 7.

7. Consider the model

$$x_t = \begin{cases} \alpha^{(1)} + \phi^{(1)} x_{t-1} + w_t^{(1)}, & \text{if } x_{t-1} < 0, \\ \alpha^{(2)} + \phi^{(2)} x_{t-1} + w_t^{(2)}, & \text{if } x_{t-1} \geq 0, \end{cases}$$

where $\{w_t^{(1)}\}$ and $\{w_t^{(2)}\}$ are independent white noise processes.

Describe how to reformulate this model in such a way that the parameters $\alpha^{(1)}$, $\alpha^{(2)}$, $\phi^{(1)}$ and $\phi^{(2)}$ may be estimated by standard methods (like least squares).

(5p)

Solution: Let $\delta = 1$ if $x_{t-1} < 0$ and 0 otherwise.

Then, the model may be written as

$$x_t = \alpha^{(1)}\delta + \phi^{(1)}\delta x_{t-1} + \alpha^{(2)}(1-\delta) + \phi^{(2)}(1-\delta)x_{t-1} + w_t,$$

where $w_t = \delta w_t^{(1)} + (1-\delta) w_t^{(2)}$ is a white noise process with variance $\delta^2\sigma_1^2 + (1-\delta)^2\sigma_2^2$, with σ_1^2, σ_2^2 as the variances of $\{w_t^{(1)}\}$ and $\{w_t^{(2)}\}$, respectively.

This is a regression model with response x_t and covariates δ, δx_{t-1}, $(1-\delta)$ and $(1-\delta)x_{t-1}$, that are all given from data (discarding the first observation for the response x_t), and without intercept.

The parameters $\alpha^{(1)}$, $\alpha^{(2)}$, $\phi^{(1)}$ and $\phi^{(2)}$ of this model may be readily estimated by standard regression procedures in e.g. R.

Appendix: figures

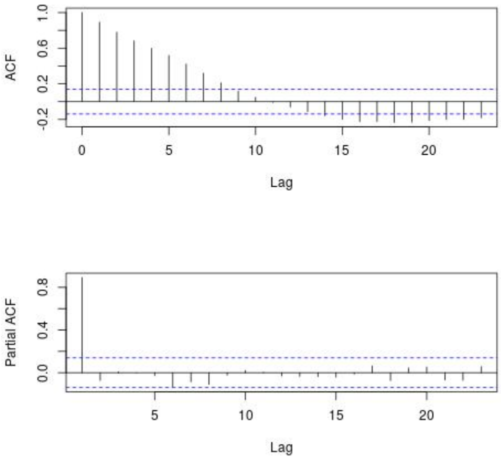

Figure 1: ACF and PACF, problem 6.

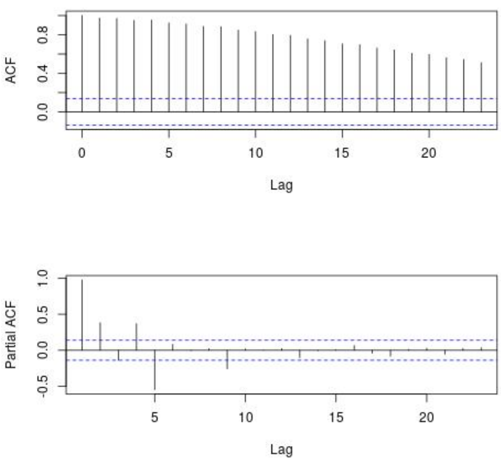

Figure 2: ACF and PACF, problem 6.

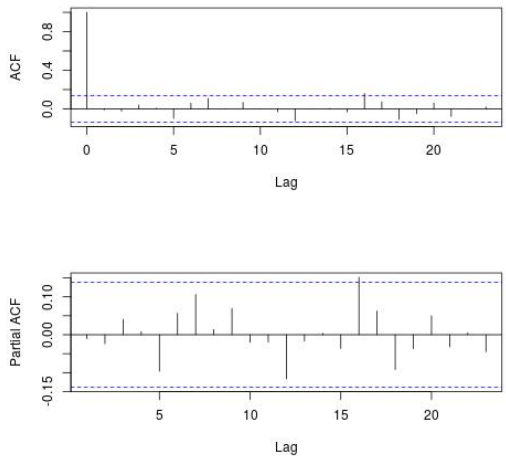

Figure 3: ACF and PACF, problem 6.

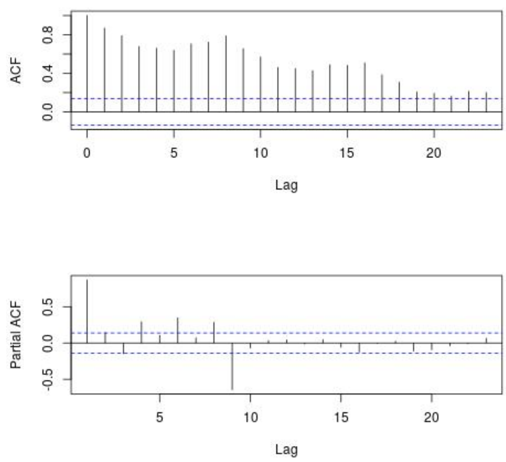

Figure 4: ACF and PACF, problem 6.

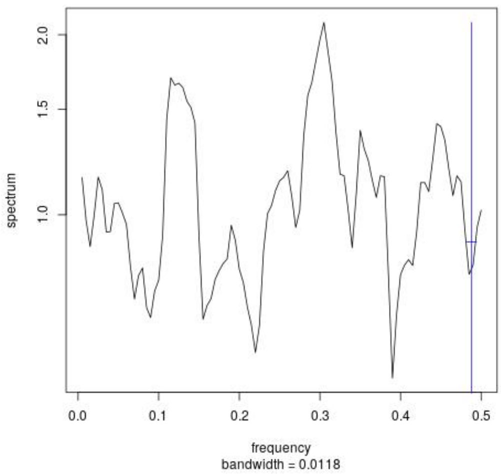

Figure 5: Estimated spectral density, problem 6.

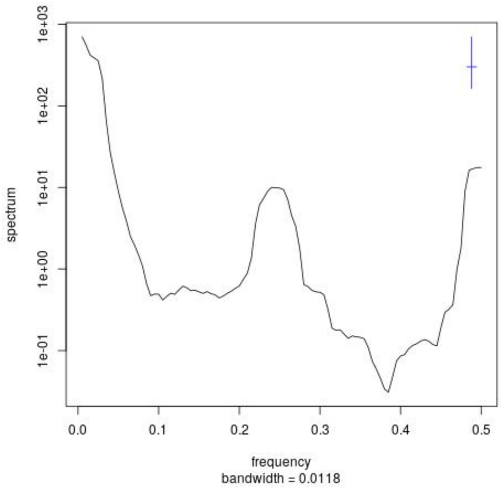

Figure 6: Estimated spectral density, problem 6.

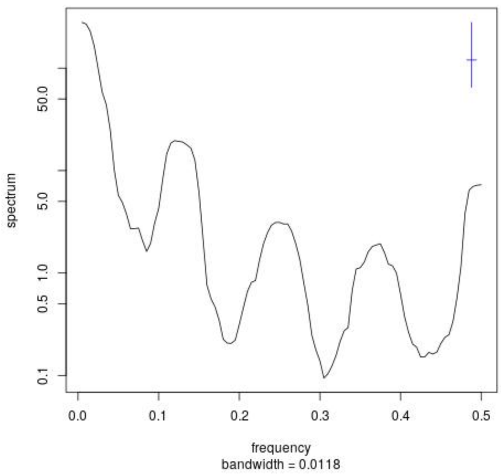

Figure 7: Estimated spectral density, problem 6.

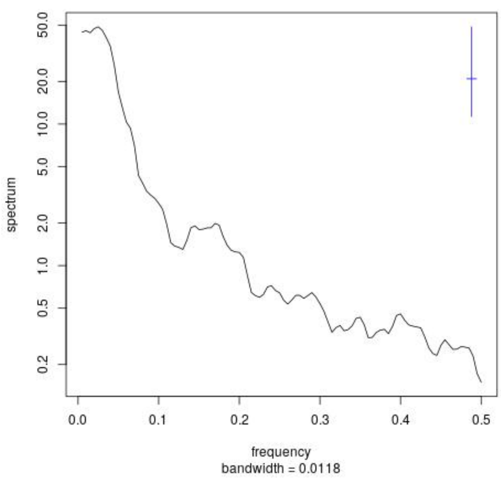

Figure 8: Estimated spectral density, problem 6.