UPPSALA UNIVERSITY
Department of Mathematics
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Exam in Mathematical Statistics
Analysis of Time Series 1MS014
2024–05–24

Time: 14.00-19.00. Limits for the credits 3, 4, 5 are 18, 25 and 32 points, respectively, including bonus points. The solutions should be well motivated.

Permitted aids: The course book or copies thereof. Hand-written sheet of formulae. Pocket calculator. Dictionary. *No electronic device with internet connection.*

1. Let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 2$ and let
$$x_t = 1 + 2t + 0.5w_t w_{t-1} w_{t-2} + 0.4w_{t-1} w_{t-2} w_{t-3}.$$
Calculate the mean and autocovariance function of x_t and state whether it is weakly stationary. (5p)

2. For the ARMA(p, q) models below, where $\{w_t\}$ are white noise processes, find p and q and determine whether they are causal and/or invertible. (6p)
   (a) $x_t = 0.4x_{t-1} + w_t - 0.4w_{t-1}$
   (b) $x_t = 0.5x_{t-1} + w_t + 0.5w_{t-1}$
   (c) $x_t = 1.5x_{t-1} + x_{t-2} + w_t$
   (d) $x_t = x_{t-1} - 0.5x_{t-2} + w_t - w_{t-1}$

3. Let $\{w_t\}$ be a white noise process with variance σ_w^2 and define the stationary process x_t through
$$x_t = \phi_1 x_{t-1} + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2}.$$
We have observations $x_1, ..., x_{100}$. The maximum likelihood estimates of the parameters are given by a computer program as $\hat{\phi}_1 = 0.45$, $\hat{\theta}_1 = 0.20$, $\hat{\theta}_2 = 0.15$ and $\hat{\sigma}_w^2 = 2.5$.
   (a) Calculate approximate standard errors of $\hat{\phi}_1$, $\hat{\theta}_1$ and $\hat{\theta}_2$. (4p)
   (b) Calculate a 95% confidence interval for ϕ_1. (2p)

4. Consider the process
$$x_t = 0.9x_{t-1} - 0.2x_{t-2} + w_t + 0.5w_{t-1}$$
where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 0.2$. We observe x_t up to time $t = 200$, where the last four observations are $x_{197} = 0.2$, $x_{198} = 0.2$, $x_{199} = 0.1$ and $x_{200} = 0.1$.
   (a) Predict the values of x_{201} and x_{202}. Approximations are permitted. (4p)
   (b) Calculate 95% prediction intervals for x_{201} and x_{202}. (2p)

5. A time series $\{x_t\}$ follows the model

$$x_t = (1 - 0.5B)(1 + 0.8B^4)w_t,$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 1$.

(a) Calculate the autocovariance function. (2p)

(b) Calculate the spectral density of x_t at the frequency $\omega = 0.25$. (2p)

(c) Let

$$y_t = \frac{1}{4}(x_t + x_{t-1} + x_{t-2} + x_{t-3}).$$

Calculate the spectral density of y_t at the frequency $\omega = 0.25$ and discuss your results. (2p)

6. Four time series of length 200 were generated. Their estimated ACF and PACF are given in figures 1-4 below. Figures 5-7, given in a "random" order, in turn depict the estimated spectral densities for three of them (spans=8).

Match three of the figures 1-4 with figures 5-7. Motivate your answer. (5p)

7. Consider the GARCH model

$$y_t = \sigma_t \epsilon_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

where the ϵ_t are i.i.d. $N(0,1)$.

(a) Show that $E(y_t) = 0$. (1p)

(b) Show that if $\alpha_1 + \beta_1 < 1$, then

$$\text{Var}(y_t) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}.$$

Without proof, you may assume that y_t is stationary. (2p)

(c) Let $\beta_1 = 0$ and assume that $|\alpha_1| < 1/\sqrt{3}$.
Derive a formula for $E(y_t^4)$ as a function of α_0 and α_1.
You may use without proof that $E(\epsilon_t^4) = 3$. (3p)

GOOD LUCK!

Appendix: figures

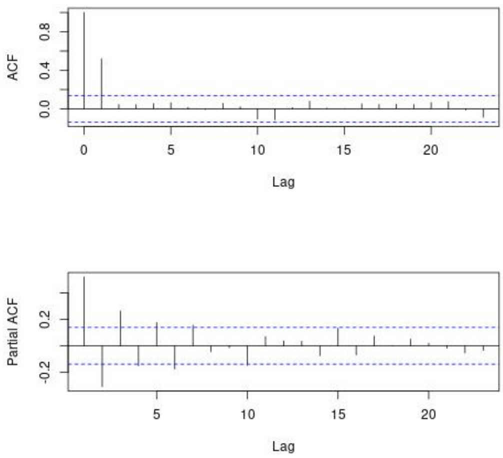

Figure 1: ACF and PACF for one of the series in problem 6.

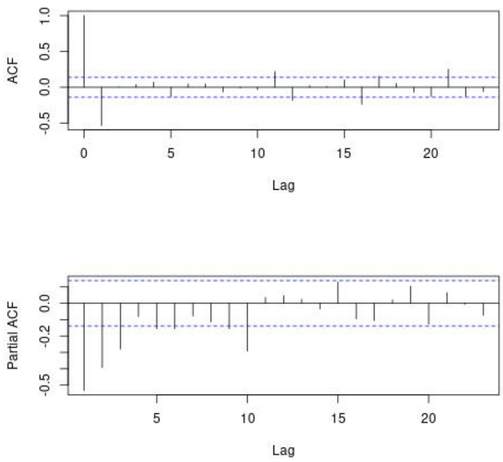

Figure 2: ACF and PACF for one of the series in problem 6.

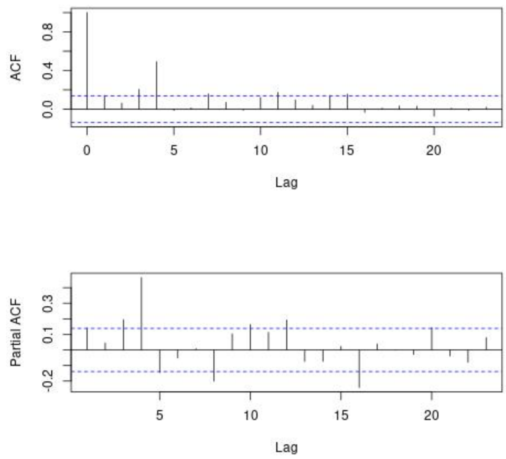

Figure 3: ACF and PACF for one of the series in problem 6.

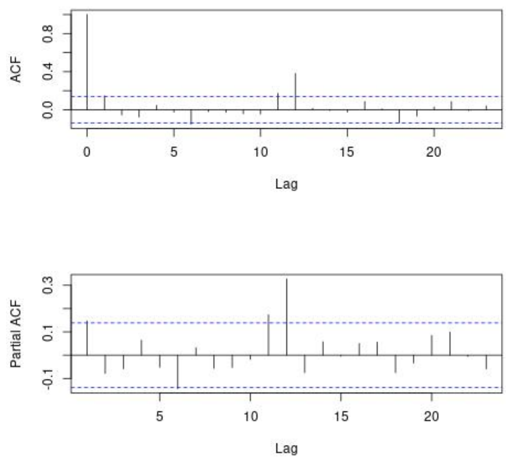

Figure 4: ACF and PACF for one of the series in problem 6.

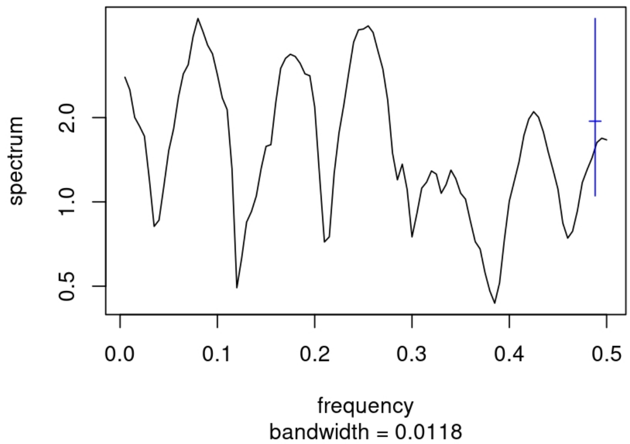

Figure 5: Estimated spectral density, problem 6.

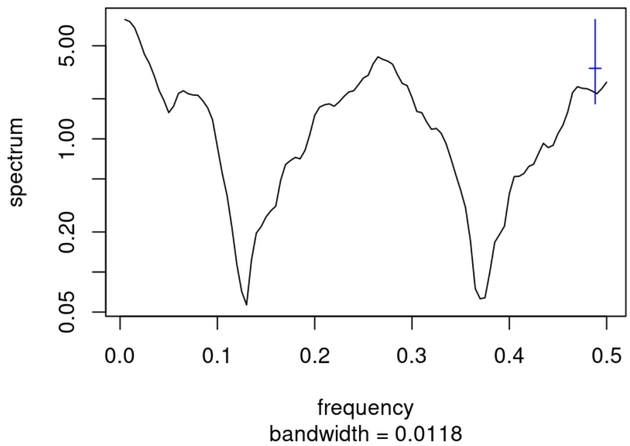

Figure 6: Estimated spectral density, problem 6.

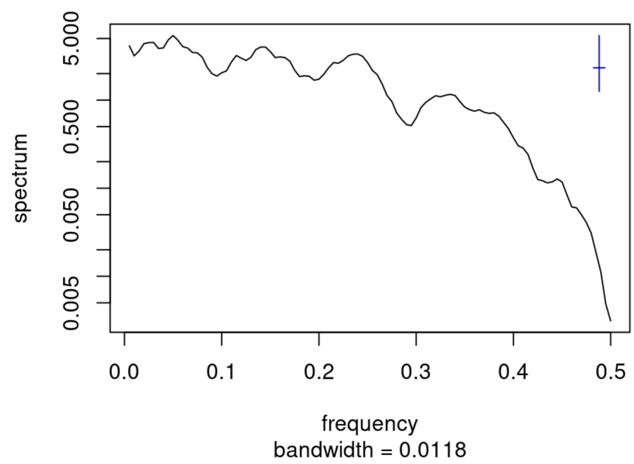

Figure 7: Estimated spectral density, problem 6.