UPPSALA UNIVERSITY
Department of Mathematics
Rolf Larsson

Exam in Mathematical Statistics
Analysis of Time Series 1MS014
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Solutions

1. Let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 2$ and let
$$x_t = 1 + 2t + 0.5w_t w_{t-1} w_{t-2} + 0.4w_{t-1} w_{t-2} w_{t-3}.$$
Calculate the mean and autocovariance function of x_t and state whether it is weakly stationary. (5p)

Solution: Because w_s and w_t are independent for $s \neq t$, the mean function is given by
$$\begin{aligned}\mu_t = E(x_t) &= 1 + 2t + 0.5E(w_t w_{t-1} w_{t-2}) + 0.4E(w_{t-1} w_{t-2} w_{t-3}) \\ &= 1 + 2t + 0.5E(w_t)E(w_{t-1})E(w_{t-2}) + 0.4E(w_{t-1})E(w_{t-2})E(w_{t-3}) \\ &= 1 + 2t.\end{aligned}$$
Using
$$x_t - \mu_t = 0.5w_t w_{t-1} w_{t-2} + 0.4w_{t-1} w_{t-2} w_{t-3},$$
the autocovariance function is found from
$$\begin{aligned}&\gamma(t+h, t) \\ &= \text{cov}(x_{t+h}, x_t) = E\{(x_{t+h} - \mu_{t+h})(x_t - \mu_t)\} \\ &= E\{(0.5w_{t+h} w_{t+h-1} w_{t+h-2} + 0.4w_{t+h-1} w_{t+h-2} w_{t+h-3}) \\ &\qquad \cdot (0.5w_t w_{t-1} w_{t-2} + 0.4w_{t-1} w_{t-2} w_{t-3})\} \\ &= 0.5^2 E(w_{t+h} w_{t+h-1} w_{t+h-2} w_t w_{t-1} w_{t-2}) \\ &+ 0.5 \cdot 0.4E(w_{t+h} w_{t+h-1} w_{t+h-2} w_{t-1} w_{t-2} w_{t-3}) \\ &+ 0.5 \cdot 0.4E(w_{t+h-1} w_{t+h-2} w_{t+h-3} w_t w_{t-1} w_{t-2}) \\ &+ 0.4^2 E(w_{t+h-1} w_{t+h-2} w_{t+h-3} w_{t-1} w_{t-2} w_{t-3}).\end{aligned}$$
Here, similar to above we have e.g. that
$$\begin{aligned}&E(w_{t+h} w_{t+h-1} w_{t+h-2} w_t w_{t-1} w_{t-2}) \\ &= E(w_t^2)E(w_{t-1}^2)E(w_{t-2}^2)I\{h = 0\} = 2^3 I\{h = 0\},\end{aligned}$$
where $I\{A\} = 1$ if A is fulfilled and 0 otherwise, and we get
$$\begin{aligned}&\gamma(t+h, t) \\ &= 2^3 \cdot (0.25I\{h = 0\} + 0.2I\{h = -1\} + 0.2I\{h = 1\} + 0.16I\{h = 0\}) \\ &= 3.28I\{h = 0\} + 1.6I\{|h| = 1\} \\ &= \begin{cases} 3.28 & \text{if } h = 0, \\ 1.6 & \text{if } |h| = 1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$
Because μ_t is a function of t, x_t is not weakly stationary.

2. For the ARMA(p, q) models below, where $\{w_t\}$ are white noise processes, find p and q and determine whether they are causal and/or invertible. (6p)

   (a) $x_t = 0.4x_{t-1} + w_t - 0.4w_{t-1}$

   Solution: We may write this model as

   $$(1 - 0.4B)x_t = (1 - 0.4B)w_t,$$

   from which it follows by cancellation of $1 - 0.4B$ on both sides that $x_t = w_t$. Hence, we have a white noise process, i.e. $p = q = 0$.
   White noise is both casual and invertible.

   (b) $x_t = 0.5x_{t-1} + w_t + 0.5w_{t-1}$

   Solution: This is an ARMA model $\phi(B)x_t = \theta(B)w_t$ with $\phi(B) = 1-0.5B$ and $\theta(B) = 1 + 0.5B$. The roots of $\phi(z)$ and $\theta(z)$ are distinct, so this is an ARMA(1,1) model, i.e. $p = q = 1$.
   Since $0 = \phi(z) = 1 - 0.5z$ has the solution $z = 2$ and $|2| = 2 > 1$, the model is causal. It is also invertible, since $0 = \theta(z) = 1 + 0.5z$ is solved by $z = -2$, and $|-2| = 2 > 1$.

   (c) $x_t = 1.5x_{t-1} + x_{t-2} + w_t$

   Solution: This is clearly seen to be an AR(2) model, i.e. $p = 2$, $q = 0$. It is invertible, since all AR models are. To see if it is causal, we check the roots of

   $$0 = \phi(z) = 1 - 1.5z - z^2,$$

   i.e. $z^2 + (3/2)z - 1 = 0$, which are given by

   $$z_{1,2} = -\frac{3}{4} \pm \sqrt{\left(\frac{3}{4}\right)^2 + 1} = -\frac{3}{4} \pm \sqrt{\frac{25}{16}} = \frac{-3 \pm 5}{4},$$

   i.e. $z_1 = 1/2$ and $z_2 = -2$. Because $|z_1| < 1$, there is one root inside the complex unit circle, and so, the model is not causal.

   (d) $x_t = x_{t-1} - 0.5x_{t-2} + w_t - w_{t-1}$

   Solution: We may write the model as $\phi(B)x_t = \theta(B)w_t$ with $\phi(B) = 1 - B + 0.5B^2$ and $\theta(B) = 1 - B$. $0 = \theta(z) = 1 - z$ gives the root $z = 1$, and since $\phi(1) \neq 0$, there is no common root. Hence, we have an ARMA(2,1) model, i.e. $p = 2$, $q = 1$.
   The MA polynomial root is on the complex unit circle, and so the model is non invertible. Is it causal? We need to solve

   $$0 = \phi(z) = 1 - z + \frac{1}{2}z^2,$$

   i.e. $z^2 - 2z + 2$, which has the roots

   $$z_{1,2} = 1 \pm \sqrt{1 - 2} = 1 \pm \sqrt{-1} = 1 \pm i.$$

   Hence, $|z_{1,2}|^2 = 2 > 1$. Both roots are outside the complex unit circle, implying that the model is causal.

3. Let $\{w_t\}$ be a white noise process with variance σ_w^2 and define the stationary process x_t through

$$x_t = \phi_1 x_{t-1} + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2}.$$

We have observations $x_1, ..., x_{100}$. The maximum likelihood estimates of the parameters are given by a computer program as $\hat{\phi}_1 = 0.45$, $\hat{\theta}_1 = 0.20$, $\hat{\theta}_2 = 0.15$ and $\hat{\sigma}_w^2 = 2.5$.

(a) Calculate approximate standard errors of $\hat{\phi}_1$, $\hat{\theta}_1$ and $\hat{\theta}_2$. (4p)

Solution: This problem became more difficult than intended! Write $\phi(B)x_t = \theta(B)w_t$, where $\phi(B) = 1 - \phi_1 B$ and $\theta(B) = 1 + \theta_1 B + \theta_2 B^2$. The sample size $n = 100$ may be thought of as large. From Property 3.10 in the course book, the asymptotic covariance matrix of the estimators $(\hat{\phi}_1, \hat{\theta}_1, \hat{\theta}_2)'$ is given by $n^{-1}\sigma_w^2\Gamma^{-1}$, where

$$\Gamma = \begin{pmatrix} \Gamma_{\phi\phi} & \Gamma_{\phi\theta} \\ \Gamma_{\theta\phi} & \Gamma_{\theta\theta} \end{pmatrix}, \tag{1}$$

with

$$\Gamma_{\phi\phi} = \sigma_w^2/(1-\phi_1^2), \tag{2}$$

as for $\phi(B)y_t = w_t$ and

$$\Gamma_{\theta\theta} = \begin{pmatrix} 1+\theta_2 & -\theta_1 \\ -\theta_1 & 1+\theta_2 \end{pmatrix} \frac{\sigma_w^2}{(1-\theta_2)\{(1+\theta_2)^2 - \theta_1^2\}}, \tag{3}$$

as for $\theta(B)z_t = w_t$, see the book. Moreover, we have

$$\Gamma_{\theta\phi} = \begin{pmatrix} \gamma_{zy}(0) \\ \gamma_{zy}(1) \end{pmatrix} = \Gamma'_{\phi\theta}. \tag{4}$$

Noting that

$$\mathrm{cov}(z_t, w_t) = \mathrm{cov}(-\theta_1 z_{t-1} - \theta_2 z_{t-2} + w_t, w_t) = \sigma_w^2,$$

we may derive the equations

$$\begin{aligned}
\gamma_{zy}(0) &= \mathrm{cov}(-\theta_1 z_{t-1} - \theta_2 z_{t-2} + w_t, \phi_1 y_{t-1} + w_t) \\
&= -\phi_1\theta_1\gamma_{zy}(0) - \phi_1\theta_2\gamma_{zy}(-1) + \sigma_w^2, \\
\gamma_{zy}(1) &= \mathrm{cov}(-\theta_1 z_t - \theta_2 z_{t-1} + w_{t+1}, \phi_1 y_{t-1} + w_t) \\
&= -\phi_1\theta_1\gamma_{zy}(1) - \phi_1\theta_2\gamma_{zy}(0) - \theta_1\sigma_w^2, \\
\gamma_{zy}(1) &= \mathrm{cov}(-\theta_1 z_t - \theta_2 z_{t-1} + w_{t+1}, y_t) \\
&= -\theta_1\gamma_{zy}(0) - \theta_2\gamma_{zy}(-1),
\end{aligned}$$

which we may rewrite as

$$\begin{aligned}
(1+\phi_1\theta_1)\gamma_{zy}(0) &= -\phi_1\theta_2\gamma_{zy}(-1) + \sigma_w^2, &(5)\\
(1+\phi_1\theta_1)\gamma_{zy}(1) &= -\phi_1\theta_2\gamma_{zy}(0) - \theta_1\sigma_w^2, &(6)\\
\gamma_{zy}(1) &= -\theta_1\gamma_{zy}(0) - \theta_2\gamma_{zy}(-1). &(7)
\end{aligned}$$

From (7), we have $-\theta_2\gamma_{zy}(-1) = \gamma_{zy}(1) + \theta_1\gamma_{zy}(0)$, which we may insert into (5) to get

$$\gamma_{zy}(0) = \phi_1\gamma_{zy}(1) + \sigma_w^2. \tag{8}$$

Next, inserting (8) into (6), we obtain

$$\gamma_{zy}(1) = -\frac{\phi_1\theta_2 + \theta_1}{1 + \phi_1\theta_1 + \phi_1^2\theta_2}\sigma_w^2,$$

which from (8) yields

$$\gamma_{zy}(0) = -\phi_1\frac{\phi_1\theta_2 + \theta_1}{1 + \phi_1\theta_1 + \phi_1^2\theta_2}\sigma_w^2 + \sigma_w^2 = \frac{\sigma_w^2}{1 + \phi_1\theta_1 + \phi_1^2\theta_2}.$$

Now, plugging in the parameter estimates into (1)-(4), we obtain

$$(\hat{\sigma}_w^2)^{-1}\hat{\Gamma} = \begin{pmatrix} 1.2539 & 0.8926 & -0.2388 \\ 0.8926 & 1.0549 & -0.1835 \\ -0.2388 & -0.1835 & 1.0549 \end{pmatrix},$$

from which we get the diagonal elements of the inverse $\hat{\sigma}_w^2\hat{\Gamma}^{-1}$ as 2.0329, 2.3843, 0.9911, giving the standard errors for $\hat{\phi}_1$, $\hat{\theta}_1$ and $\hat{\theta}_2$ respectively as $\sqrt{2.039/100} \approx 0.1426$, $\sqrt{2.3843/100} \approx 0.1544$ and $\sqrt{0.9911/100} \approx 0.0996$.

(b) Calculate a 95% confidence interval for ϕ_1. (2p)

Solution: Because of asymptotic normality, a confidence interval for ϕ_1 with approximate 95% confidence level is given by the estimate $\pm$ 1.96 times the standard error as

$$0.45 \pm 1.96 \cdot 0.1426 = 0.45 \pm 0.28 = (0.17,\ 0.73).$$

4. Consider the process

$$x_t = 0.9x_{t-1} - 0.2x_{t-2} + w_t + 0.5w_{t-1}$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 0.2$. We observe x_t up to time $t = 200$, where the last four observations are $x_{197} = 0.2$, $x_{198} = 0.2$, $x_{199} = 0.1$ and $x_{200} = 0.1$.

(a) Predict the values of x_{201} and x_{202}. Approximations are permitted. (4p)

Solution: We will calculate truncated predictions by using the AR representation $\pi(B)x_t = w_t$. We have

$$(1 + 0.5B)w_t = (1 - 0.9B + 0.2B^2)x_t,$$

which yields

$$\pi(B)(1 + 0.5B)w_t = (1 - 0.9B + 0.2B^2)\pi(B)x_t = (1 - 0.9B + 0.2B^2)w_t.$$

Hence, with $\pi(z) = 1 + \pi_1 z + \pi_2 z^2 + ...$, we need to solve

$$(1 + \pi_1 z + \pi_2 z^2 + \pi_3 z^3 + ...)(1 + 0.5z) = 1 - 0.9z + 0.2z^2,$$

i.e.

$$1 + (\pi_1 + 0.5)z + (\pi_2 + 0.5\pi_1)z^2 + (\pi_3 + 0.5\pi_2)z^3 + ... = 1 - 0.9z + 0.2z^2,$$

which yields

$$\begin{aligned}
\pi_1 &= -0.5 - 0.9 = -1.4, \\
\pi_2 &= -0.5\pi_1 + 0.2 = 0.9, \\
\pi_3 &= -0.5\pi_2 = -0.45, \\
\pi_4 &= -0.5\pi_3 = 0.225, \\
\pi_5 &= -0.5\pi_4 = -0.1125.
\end{aligned}$$

The truncated predictions become

$$\begin{aligned}
\tilde{x}_{201} &= -\pi_1 x_{200} - \pi_2 x_{199} - ... \\
&\approx -(-1.4)\cdot 0.1 - 0.9\cdot 0.1 - (-0.45)\cdot 0.2 - 0.225\cdot 0.2 \\
&= 0.095
\end{aligned}$$

and

$$\begin{aligned}
\tilde{x}_{202} &= -\pi_1 \tilde{x}_{201} - \pi_2 x_{200} - \pi_3 x_{199} - ... \\
&\approx -(-1.4)\cdot 0.095 - 0.9\cdot 0.1 - (-0.45)\cdot 0.1 - 0.225\cdot 0.2 \\
&\quad - (-0.1125)\cdot 0.2 \\
&= 0.0655.
\end{aligned}$$

(b) Calculate 95% prediction intervals for x_{201} and x_{202}. (2p)

Solution: The mean square predicion error m steps ahead is given by $\sigma_w^2 \sum_{j=1}^{m-1} \psi_j^2$, where the ψ_j are the coefficients in the MA representation, with $\psi_0 = 1$. We only need to find ψ_1. To this end, $x_t = \psi(B)w_t$ implies

$$\psi(B)(1 - 0.9B + 0.2B^2)x_t = (1 + 0.5B)\psi(B)w_t = (1 + 0.5B)x_t,$$

so that with $\psi(z) = 1 + \psi_1 z + ...$, we have

$$(1 + \psi_1 z + ...)(1 - 0.9z + 0.2z^2) = 1 + 0.5z,$$

implying $\psi_1 - 0.9 = 0.5$, i.e. $\psi_1 = 1.4$.
With $\sigma_w^2 = 0.2$, this gives the 95% prediction interval for x_{201} as

$$0.095 \pm 1.96\sqrt{0.2} = 0.095 \pm 0.877 = (-0.782,\ 0.972),$$

and for x_{202}, we find the corresponding interval

$$0.0655 \pm 1.96\sqrt{0.2(1 + 1.4^2)} = 0.0655 \pm 1.5081 = (-1.443,\ 1.574).$$

5. A time series $\{x_t\}$ follows the model

$$x_t = (1 - 0.5B)(1 + 0.8B^4)w_t,$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 1$.

(a) Calculate the autocovariance function. (2p)

Solution: We write the model as

$$\begin{aligned} x_t &= (1 - 0.5B + 0.8B^4 - 0.4B^5)w_t \\ &= w_t - 0.5w_{t-1} + 0.8w_{t-4} - 0.4w_{t-5}, \end{aligned}$$

which gives the autocovariance function as

$$\begin{aligned} \gamma(h) &= \operatorname{cov}(x_{t+h}, x_t) \\ &= \operatorname{cov}(w_{t+h} - 0.5w_{t+h-1} + 0.8w_{t+h-4} - 0.4w_{t+h-5}, \\ &\qquad w_t - 0.5w_{t-1} + 0.8w_{t-4} - 0.4w_{t-5}). \end{aligned}$$

This gives a sum of 16 terms. If $|h| = 2$ or $|h| \geq 6$, we can check that they are all zero. For other h, we can take them case by case and come up with the expression

$$\begin{aligned} \gamma(h) &= (1 + 0.5^2 + 0.8^2 + 0.4^2)I\{h = 0\} + (-0.5 - 0.8 \cdot 0.4)I\{|h| = 1\} \\ &\quad - 0.5 \cdot 0.8I\{|h| = 3\} + (0.8 + 0.5 \cdot 0.4)I\{|h| = 4\} - 0.4I\{|h| = 5\} \\ &= 2.05I\{h = 0\} - 0.82I\{|h| = 1\} - 0.4I\{|h| = 3\} + I\{|h| = 4\} \\ &\quad - 0.4I\{|h| = 5\} \\ &= \begin{cases} 2.05 & \text{if } h = 0, \\ -0.82 & \text{if } |h| = 1, \\ -0.4 & \text{if } |h| = 3 \text{ or } 5, \\ 1 & \text{if } |h| = 4, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(b) Calculate the spectral density of x_t at the frequency $\omega = 0.25$. (2p)

Solution: We may use the result from (a) and the general fomula for the spectral density to obtain

$$\begin{aligned} f(\omega) &= \sum_{h=-\infty}^{\infty} \gamma(h)e^{-2\pi i\omega h} \\ &= 2.05 - 0.82(e^{2\pi i\omega} + e^{-2\pi i\omega}) - 0.4(e^{6\pi i\omega} + e^{-6\pi i\omega}) \\ &\quad + (e^{8\pi i\omega} + e^{-8\pi i\omega}) - 0.4(e^{10\pi i\omega} + e^{-10\pi i\omega}) \\ &= 2.05 - 1.64\cos(2\pi\omega) - 0.8\cos(6\pi\omega) + 2\cos(8\pi\omega) - 0.8\cos(10\pi\omega). \end{aligned}$$

Hence,

$$\begin{aligned} f(0.25) &= 2.05 - 1.64\cos\left(\frac{\pi}{2}\right) - 0.8\cos\left(\frac{3\pi}{2}\right) + 2\cos(2\pi) - 0.8\cos\left(\frac{5\pi}{2}\right) \\ &= 2.05 - 1.64 \cdot 0 - 0.8 \cdot 0 + 2 \cdot 1 - 0.8 \cdot 0 = 4.05. \end{aligned}$$

Alternatively, one may use the formula $f(\omega) = \sigma_w^2|\theta(e^{-2\pi i\omega})|^2$.

(c) Let

$$y_t = \frac{1}{4}(x_t + x_{t-1} + x_{t-2} + x_{t-3}).$$

Calculate the spectral density of y_t at the frequency $\omega = 0.25$ and discuss your results. (2p)

Solution: Write $y_t = A(B)x_t$ where

$$A(B) = \frac{1}{4}(1 + B + B^2 + B^3).$$

We will make use of the formula

$$f_{yy}(\omega) = |A(e^{-2\pi i\omega})|^2 f_{xx}(\omega),$$

where $f_{xx}(\omega)$, $f_{yy}(\omega)$ are the spectral densities of the input and output repspectively, and where $A(e^{-2\pi i\omega})$ is the frequency response function, with $A(\cdot)$ as above. Here,

$$\begin{aligned}
&|A(e^{-2\pi i\omega})|^2 \\
&= A(e^{-2\pi i\omega})A(e^{2\pi i\omega}) \\
&= \frac{1}{16}\left(1 + e^{-2\pi i\omega} + e^{-4\pi i\omega} + e^{-6\pi i\omega}\right)\left(1 + e^{2\pi i\omega} + e^{4\pi i\omega} + e^{6\pi i\omega}\right) \\
&= \frac{1}{16}\left\{4 + 3(e^{2\pi i\omega} + e^{-2\pi i\omega}) + 2(e^{4\pi i\omega} + e^{-4\pi i\omega}) + e^{6\pi i\omega} + e^{-6\pi i\omega}\right\} \\
&= \frac{1}{16}\left\{4 + 6\cos(2\pi\omega) + 4\cos(4\pi\omega) + 2\cos(6\pi\omega)\right\}.
\end{aligned}$$

Inserting $\omega = 0.25 = 1/4$ here, we get

$$\begin{aligned}
&\frac{1}{16}\left\{4 + 6\cos\left(\frac{\pi}{2}\right) + 4\cos(\pi) + 2\cos\left(\frac{3\pi}{2}\right)\right\} \\
&= \frac{1}{16}\left\{4 + 6\cdot 0 + 4\cdot(-1) + 2\cdot 0\right\} = 0,
\end{aligned}$$

implying via (b) that

$$f_{yy}(0.25) = 0 \cdot 4.05 = 0.$$

This is not surprising, since the seasonal filter kills the corresponding seasonal frequency in the input series.

6. Four time series of length 200 were generated. Their estimated ACF and PACF are given in figures 1-4 below. Figures 5-7, given in a "random" order, in turn depict the estimated spectral densities for three of them (spans=8).

   Match three of the figures 1-4 with figures 5-7. Motivate your answer. (5p)

   Solution: We can start by looking at figures 5-7. Figure 5 exhibits peaks at roughly 0.08 and multiples thereof, so it is probably the estimated spectral density of a series with period around $12 \approx 1/0.08$. The ACF and PACF in figure 4 seem to correspond to that, since they both have peaks at lag length 12.

   In figure 6, there is a peak at 0.25, corresponding to a series with period 4. This should correspond to the ACF and PACF of figure 3, which both have peaks at lag length 4.

   Finally, figure 7 displays no seasonality. The estimated spectral density is higher for low frequencies than for high ones, and this should correspond to a series with positive correlation on low lags. Comparing the ACFs of figures 1 and 2, we find a positive lag one correlation in figure 1, but in figure 2, the lag one correlation is negative. This points at figure 1 as the best match here.

   So in summary, the figures correspond to each other according to 1-7, 3-6 and 4-5. (Figure 2 is left with no match.)

7. Consider the GARCH model

$$y_t = \sigma_t \epsilon_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

where the ϵ_t are i.i.d. $N(0,1)$.

(a) Show that $E(y_t) = 0$. (1p)

Solution: Let $Y_s = \{y_s, y_{s-1}, ...\}$. i.e. all information gathered up to time s. By the law of iterated expectations, we have

$$\begin{aligned} E(y_t) &= E\{E(y_t|Y_{t-1})\} = E\{E(\sigma_t \epsilon_t|Y_{t-1})\} = E\{\sigma_t E(\epsilon_t|Y_{t-1})\} \\ &= E\{\sigma_t E(\epsilon_t)\} = 0. \end{aligned}$$

Here, the third equality follows because σ_t is a function of Y_{t-1}, the fourth equality follows because ϵ_t is independent of Y_{t-1}, and the last equality follows since $E(\epsilon_t) = 0$.

(b) Show that if $\alpha_1 + \beta_1 < 1$, then

$$\mathrm{Var}(y_t) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}.$$

Without proof, you may assume that y_t is stationary. (2p)

Solution: At first, let $v_t = y_t^2 - \sigma_t^2$. Then,

$$\begin{aligned} v_t - \beta_1 v_{t-1} &= y_t^2 - \sigma_t^2 - \beta_1(y_{t-1}^2 - \sigma_{t-1}^2) \\ &= y_t^2 - \beta_1 y_{t-1}^2 - (\sigma_t^2 - \beta_1 \sigma_{t-1}^2) \\ &= y_t^2 - \beta_1 y_{t-1}^2 - (\alpha_0 + \alpha_1 y_{t-1}^2) \\ &= y_t^2 - \alpha_0 - (\alpha_1 + \beta_1) y_{t-1}^2, \end{aligned}$$

where the third line follows from the second model equation. Hence,

$$y_t^2 = \alpha_0 + (\alpha_1 + \beta_1) y_{t-1}^2 + v_t - \beta_1 v_{t-1}. \tag{9}$$

(Instead of this derivation, it is permitted to refer to p.258 in the course book.)
Moreover, writing $v_t = \sigma_t^2(\epsilon_t^2 - 1)$, we have as in the solution to (a) that

$$\begin{aligned} E(v_t) &= E\{E(\sigma_t^2(\epsilon_t^2 - 1)|Y_{t-1})\} = E\{\sigma_t^2 E(\epsilon_t^2 - 1|Y_{t-1})\} \\ &= E\{\sigma_t^2 E(\epsilon_t^2 - 1)\} = 0. \end{aligned}$$

Now, by the stationarity of y_t, $\mathrm{Var}(y_t)$ does not depend on t, and (a) and (9) imply

$$\begin{aligned} \mathrm{Var}(y_t) &= E(y_t^2) = \alpha_0 + (\alpha_1 + \beta_1) E(y_{t-1}^2) + E(v_t) - \beta_1 E(v_{t-1}) \\ &= \alpha_0 + (\alpha_1 + \beta_1) \mathrm{Var}(y_t), \end{aligned}$$

and solving for $\mathrm{Var}(y_t)$, we obtain

$$\mathrm{Var}(y_t) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}.$$

(c) Let $\beta_1 = 0$ and assume that $|\alpha_1| < 1/\sqrt{3}$.
Derive a formula for $E(y_t^4)$ as a function of α_0 and α_1.
You may use without proof that $E(\epsilon_t^4) = 3$. (3p)

Solution: As in (a), we have by the hint that

$$E(y_t^4|Y_{t-1}) = E(\sigma_t^4\epsilon_t^4|Y_{t-1}) = \sigma_t^4 E(\epsilon_t^4|Y_{t-1}) = \sigma_t^4 E(\epsilon_t^4) = 3\sigma_t^4,$$

and using $\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2$, we thus obtain

$$\begin{aligned} E(y_t^4) &= E\{E(y_t^4|Y_{t-1})\} = 3E\{(\alpha_0 + \alpha_1 y_{t-1}^2)^2\} \\ &= 3\{\alpha_0^2 + 2\alpha_0\alpha_1 E(y_{t-1}^2) + \alpha_1^2 E(y_{t-1}^4)\}. \end{aligned}$$

Hence, by stationarity and (b) with $\beta_1 = 0$,

$$E(y_t^4) = 3\left\{\alpha_0^2 + 2\alpha_0\alpha_1\frac{\alpha_0}{1-\alpha_1} + \alpha_1^2 E(y_t^4)\right\},$$

and solving for $E(y_t^4)$, we obtain

$$E(y_t^4) = \frac{3\left(\alpha_0^2 + 2\alpha_0\alpha_1\frac{\alpha_0}{1-\alpha_1}\right)}{1-3\alpha_1^2} = \frac{3\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}.$$

Note that the condition $|\alpha_1| < 1/\sqrt{3}$ is equivalent to $1 - 3\alpha_1^2 > 0$.

Appendix: figures

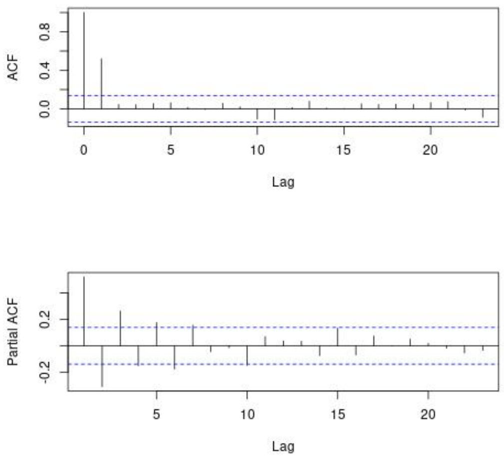

Figure 1: ACF and PACF for one of the series in problem 6.

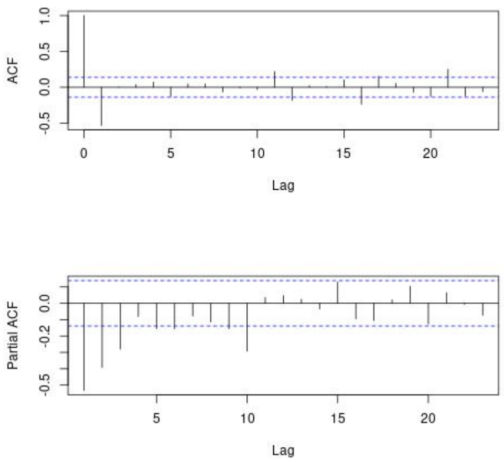

Figure 2: ACF and PACF for one of the series in problem 6.

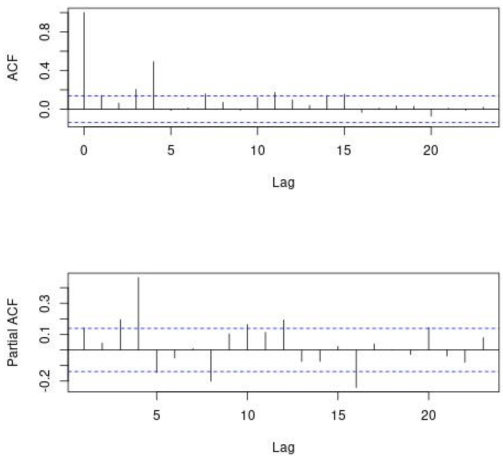

Figure 3: ACF and PACF for one of the series in problem 6.

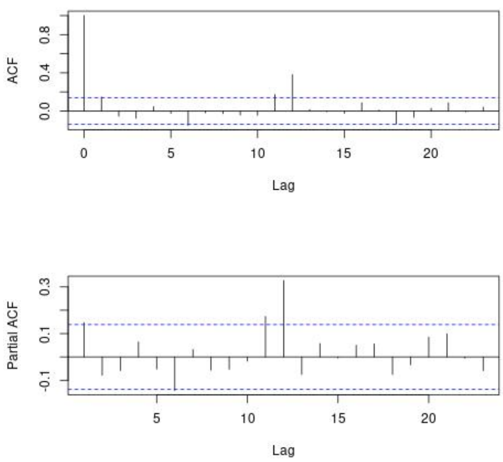

Figure 4: ACF and PACF for one of the series in problem 6.

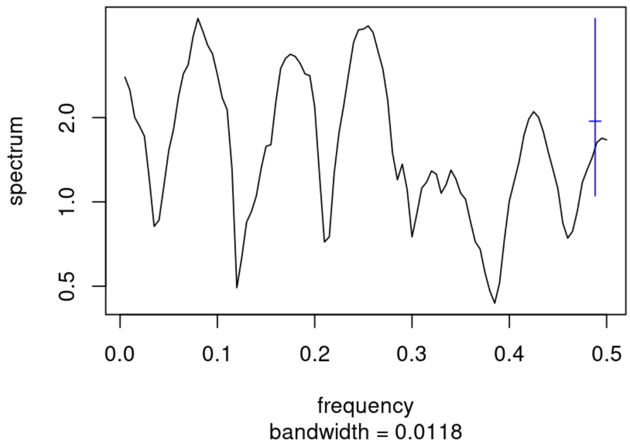

Figure 5: Estimated spectral density, problem 6.

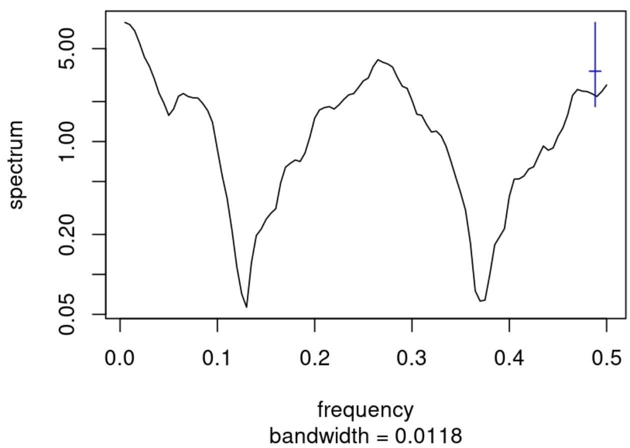

Figure 6: Estimated spectral density, problem 6.

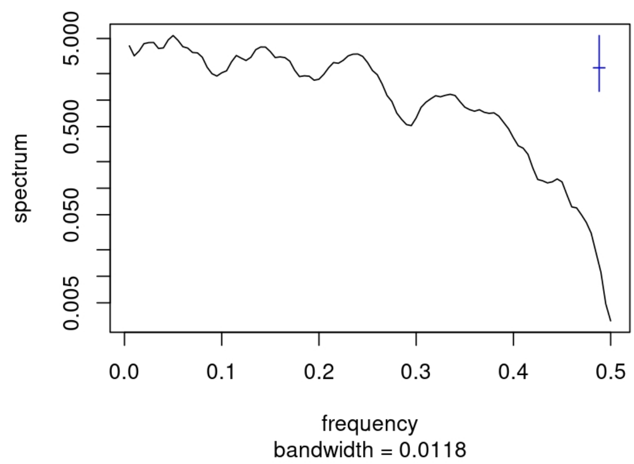

Figure 7: Estimated spectral density, problem 6.