UPPSALA UNIVERSITY
Department of Mathematics
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Exam in Mathematical Statistics
Analysis of Time Series 1MS014
2024–08–21

Time: 8.00-13.00. Limits for the credits 3, 4, 5 are 18, 25 and 32 points, respectively. The solutions should be well motivated.

Permitted aids: The course book or copies thereof. Hand-written sheet of formulae. Pocket calculator. Dictionary. *No electronic device with internet connection.*

1. Let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 2$ and let
$$x_t = 0.5w_t w_{t-1} + 0.2w_{t-2}w_{t-3}.$$
Calculate the mean and autocovariance function of x_t and state whether it is weakly stationary. (5p)

2. For the ARMA(p, q) models below, where $\{w_t\}$ are white noise processes, find p and q and determine whether they are causal and/or invertible. (6p)
   (a) $x_t = 0.2x_{t-1} + w_t + 0.2w_{t-1}$
   (b) $x_t = 0.7x_{t-1} + 0.6x_{t-2} + w_t$
   (c) $x_t = 0.7x_{t-1} + 0.6x_{t-2} + w_t - 1.2w_{t-1}$
   (d) $x_t = -0.25x_{t-2} + w_t$

3. Let $\{w_t\}$ be a white noise process with variance $\sigma_w^2 = 1$ and define x_t through
$$x_t = 0.1x_{t-2} + 0.2x_{t-4} + w_t.$$
Calculate the autocorrelation function $\rho(h)$ for $h = 1, 2, 3, 4, 5, 6$. (5p)

4. Consider the process
$$x_t = w_t + 0.3w_{t-1} - 0.1w_{t-2}$$
where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 0.16$. We observe x_t up to time $t = 200$, where the last four observations are $x_{197} = 0.8$, $x_{198} = 0.4$, $x_{199} = 0.0$ and $x_{200} = 0.4$.
   (a) Predict the values of x_{201} and x_{202}. Approximations are permitted. (4p)
   (b) Calculate 95% prediction intervals for x_{201} and x_{202}. (3p)

5. Consider the time series model

$$x_t = 0.9x_{t-4} + w_t + 0.2w_{t-1},$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 2$.

(a) Write it as a model on the form SARMA$(p, q) \times (P, Q)_s$. (2p)

(b) Calculate the spectral density of x_t at the frequency $\omega = 0.25$. (2p)

(c) Calculate the spectral density of $y_t = x_t - x_{t-4}$ at the frequency $\omega = 0.25$. (2p)

(d) Compare and discuss your results in (b) and (c). (1p)

6. Three data series were collected from the website of Statistics Sweden (SCB): The monthly number of people in the workforce 2001-2024 (Figure 1), the quarterly electrical energy balance in TJ 1984-2017 (Figure 2) and the population size 1880-2023 (Figure 3).

In Figures 4-7, estimated spectral densities of the series of Figures 1-3 are given in 'random' order, together with an estimated spectral density from another series (not shown).

Match figures 1-3 with three of the figures 4-7. Motivate your answer. (5p)

7. Consider the system

$$x_{1,t} = w_{1,t},$$
$$x_{2,t} = -0.4x_{1,t-1} + 0.5w_{1,t-1} + w_{2,t},$$

where $\{w_{1,t}\}$ and $\{w_{2,t}\}$ are white noise processes (possibly dependent of each other).

(a) Write this system on vector/matrix form as a VARMA(p, q) model. What is p and q here? (2p)

(b) Show that this model is equivalent to a VARMA$(0, 1)$ model. (3p)

GOOD LUCK!

Appendix: figures

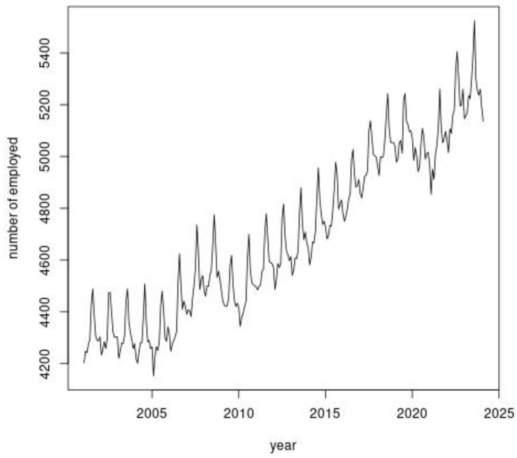

Figure 1: The monthly number of people in the workforce.

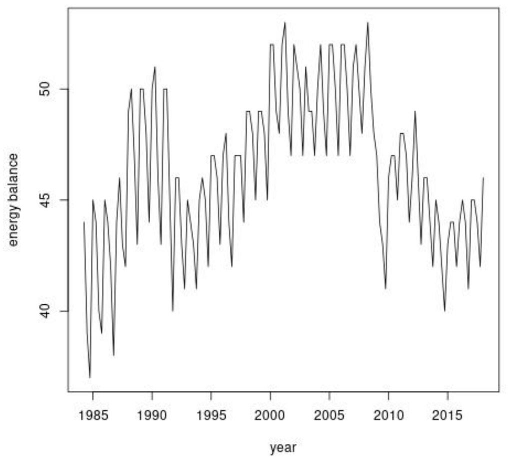

Figure 2: The quarterly electrical energy balance.

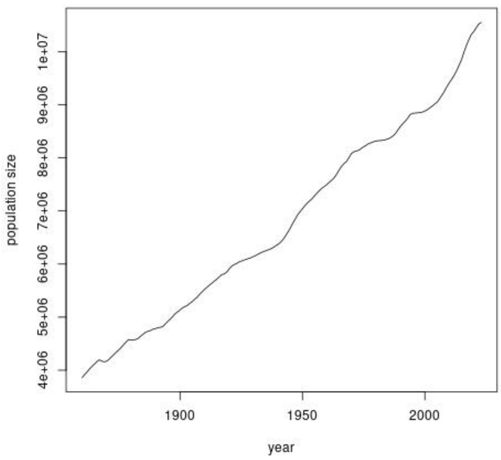

Figure 3: The population size.

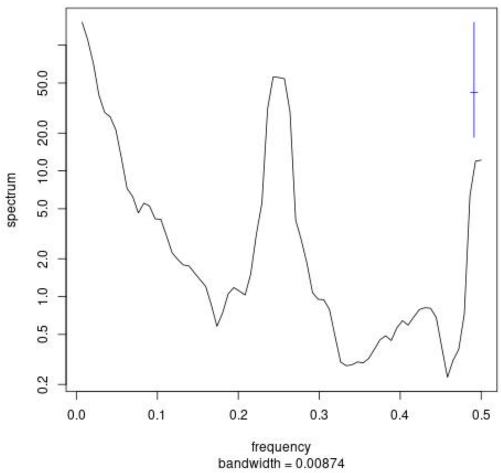

Figure 4: Estimated spectral density, problem 6.

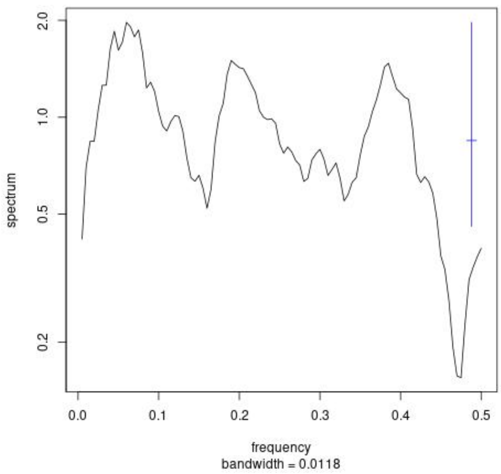

Figure 5: Estimated spectral density, problem 6.

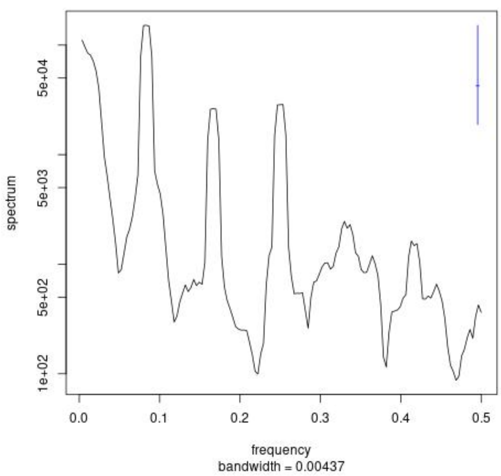

Figure 6: Estimated spectral density, problem 6.

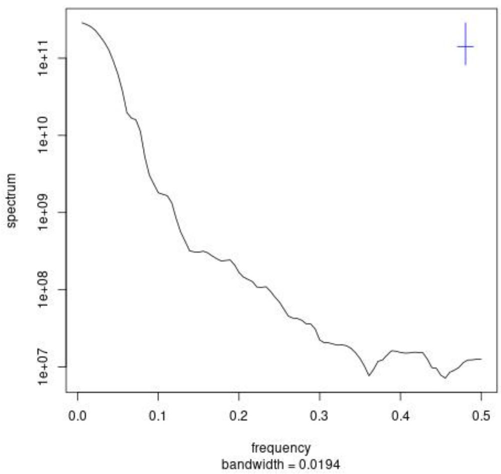

Figure 7: Estimated spectral density, problem 6.