UPPSALA UNIVERSITY
Department of Mathematics
Rolf Larsson

Exam in Mathematical Statistics
Analysis of Time Series 1MS014
2024–08–21
Solutions

1. Let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 2$ and let
$$x_t = 0.5w_t w_{t-1} + 0.2w_{t-2}w_{t-3}.$$
Calculate the mean and autocovariance function of x_t and state whether it is weakly stationary. (5p)

Solution: By independence of the $\{w_t\}$ series, the mean function is given by
$$\mu_t = E(x_t) = 0.5E(w_t)E(w_{t-1}) + 0.2E(w_{t-2})E(w_{t-3}) = 0.$$
Moreover, the autocovariance function may be written as
$$\begin{aligned}
&\gamma(t+h,t) \\
&= \text{cov}(x_{t+h}, x_t) \\
&= \text{cov}(0.5w_{t+h}w_{t+h-1} + 0.2w_{t+h-2}w_{t+h-3}, 0.5w_t w_{t-1} + 0.2w_{t-2}w_{t-3}) \\
&= 0.25\text{cov}(w_{t+h}w_{t+h-1}, w_t w_{t-1}) + 0.1\text{cov}(w_{t+h}w_{t+h-1}, w_{t-2}w_{t-3}) \\
&+ 0.1\text{cov}(w_{t+h-2}w_{t+h-3}, w_t w_{t-1}) + 0.04\text{cov}(w_{t+h-2}w_{t+h-3}, w_{t-2}w_{t-3}).
\end{aligned} \tag{1}$$
Here, by independence,
$$E(w_{t+h}w_{t+h-1}) = E(w_{t+h})E(w_{t+h-1}) = 0$$
for all h, and since $E(w_t^2) = \text{var}(w_t) = 2$, we have
$$\text{cov}(w_t w_{t-1}, w_t w_{t-1}) = E(w_t^2)E(w_{t-1}^2) = 4,$$
$$\text{cov}(w_{t+1}w_t, w_t w_{t-1}) = E(w_{t+1})E(w_t^2)E(w_{t-1}) = 0,$$
and for $h \geq 2$,
$$\text{cov}(w_{t+h}w_{t+h-1}, w_t w_{t-1}) = E(w_{t+h})E(w_{t+h-1})E(w_t)E(w_{t-1}) = 0.$$
Hence, with $I\{A\} = 1$ if A is fulfilled and 0 otherwise, (1) yields
$$\begin{aligned}
&\gamma(t+h,t) \\
&= 0.25 \cdot 4I\{h=0\} + 0.1 \cdot 4I\{h=-2\} + 0.1 \cdot 4I\{h=2\} + 0.04 \cdot 4I\{h=0\} \\
&= 1.16I\{h=0\} + 0.4I\{|h|=2\} \\
&= \begin{cases} 1.16 & \text{if } h = 0, \\ 0.4 & \text{if } |h| = 2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$
Since the variance $\gamma(t,t) = 1.16$ is finite and none of μ_t or $\gamma(t+h,t)$ is a function of t, we conclude that x_t is weakly stationary.

2. For the ARMA(p, q) models below, where $\{w_t\}$ are white noise processes, find p and q and determine whether they are causal and/or invertible. (6p)

   (a) $x_t = 0.2x_{t-1} + w_t + 0.2w_{t-1}$

   Solution: Writing the model with operators, it is

   $$(1 - 0.2B)x_t = (1 + 0.2B)w_t,$$

   This is an ARMA(1,1) model, i.e. $p = 1$ and $q = 1$. The root of the AR polynomial $1 - 0.2z$ fulfills $|z| = 5 > 1$, hence the model is causal. The root of the MA polynomial $1 + 0.2z$ fulfills $|z| = 5 > 1$, hence the model is invertible.

   (b) $x_t = 0.7x_{t-1} + 0.6x_{t-2} + w_t$

   Solution: The model is

   $$(1 - 0.7B - 0.6B^2)x_t = w_t,$$

   so it is an AR(2) model ($p = 2$, $q = 0$). Because it is a pure AR model, it is invertible.
   To see if it is causal, we solve $1 - 0.7z - 0.6z^2$, i.e.

   $$z^2 + \frac{7}{6}z - \frac{5}{3} = 0,$$

   which has the solutions

   $$z_{1,2} = -\frac{7}{12} \pm \sqrt{\frac{7^2}{12^2} + \frac{5}{3}} = \frac{-7 \pm 17}{12},$$

   i.e. $z_1 = -2$ and $z_2 = 5/6$. Here, $|z_2| < 1$ and so, the model is *not* causal.

   (c) $x_t = 0.7x_{t-1} + 0.6x_{t-2} + w_t - 1.2w_{t-1}$

   Solution: At first, the model appears to be ARMA(2,1), with the representation

   $$(1 - 0.7B - 0.6B^2)x_t = (1 - 1.2B)w_t,$$

   but because of the results of (b) and the factorization theorem of polynomials, we have

   $$(1 - 1.2B)(1 + 0.5B)x_t = (1 - 1.2B)w_t,$$

   i.e. $(1 + 0.5B)x_t = w_t$, hence an AR(1) model ($p = 1$, $q = 0$).
   Being a pure AR model, it is invertible, and it is also causal because the root of $1 + 0.5z$ fulfills $|z| = 2 > 1$.

   (d) $x_t = -0.25x_{t-2} + w_t$

   Solution: This is an AR(2) model, i.e. $p = 2$, $q = 0$, with representation

   $$(1 + 0.25B^2)x_t = w_t.$$

   It is invertible, since it is pure AR. The roots of $1 + 0.25z^2$ fulfill
   $|z| = 2 > 1$, hence it is also causal.

3. Let $\{w_t\}$ be a white noise process with variance $\sigma_w^2 = 1$ and define x_t through

$$x_t = 0.1x_{t-2} + 0.2x_{t-4} + w_t.$$

Calculate the autocorrelation function $\rho(h)$ for $h = 1, 2, 3, 4, 5, 6$. (5p)

Solution: For $h > 0$, we have

$$\begin{aligned}\gamma(h) &= \text{cov}(x_{t+h}, x_t) = \text{cov}(0.1x_{t+h-2} + 0.2x_{t+h-4} + w_{t+h}, x_t) \\ &= 0.1\text{cov}(x_{t+h-2}, x_t) + 0.2\text{cov}(x_{t+h-4}, x_t) + \text{cov}(w_{t+h}, x_t) \\ &= 0.1\gamma(h-2) + 0.2\gamma(h-4),\end{aligned}$$

and dividing through by $\gamma(0)$ and using $\rho(h) = \gamma(h)/\gamma(0)$, we obtain

$$\rho(h) = 0.1\rho(h-2) + 0.2\rho(h-4), \quad h = 1, 2, ... \tag{2}$$

We start with even h. Using $\rho(0) = 1$ and $\rho(-h) = \rho(h)$, $h = 2$ in (2) yields

$$\rho(2) = 0.1\rho(0) + 0.2\rho(-2) = 0.1 + 0.2\rho(2),$$

implying $\rho(2) = 0.1/0.8 = 1/8 = 0.125$.
Consequently, $h = 4$ in (2) gives

$$\rho(4) = 0.1\rho(2) + 0.2\rho(0) = 0.1 \cdot 0.125 + 0.2 = 0.2125,$$

and $h = 6$ in (2) gives

$$\rho(6) = 0.1\rho(4) + 0.2\rho(2) = 0.1 \cdot 0.2125 + 0.2 \cdot 0.125 = 0.04625.$$

As for odd h, at first $h = 1$ in (2) yields

$$\rho(1) = 0.1\rho(-1) + 0.2\rho(-3) = 0.1\rho(1) + 0.2\rho(3),$$

implying $\rho(3) = 4.5\rho(1)$. On the other hand, $h = 3$ in (2) gives

$$\rho(3) = 0.1\rho(1) + 0.2\rho(-1) = 0.3\rho(1),$$

which is a contradiction unless $\rho(1) = \rho(3) = 0$. Finally, $h = 5$ in (2) gives

$$\rho(5) = 0.1\rho(3) + 0.2\rho(1) = 0.$$

To sum up, the answer is $\rho(1) = 0$, $\rho(2) = 0.125$, $\rho(3) = 0$, $\rho(4) = 0.2125$, $\rho(5) = 0$ and $\rho(6) = 0.04625$.

4. Consider the process

$$x_t = w_t + 0.3w_{t-1} - 0.1w_{t-2}$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 0.16$. We observe x_t up to time $t = 200$, where the last four observations are $x_{197} = 0.8$, $x_{198} = 0.4$, $x_{199} = 0.0$ and $x_{200} = 0.4$.

(a) Predict the values of x_{201} and x_{202}. Approximations are permitted. (4p)

Solution: We will calculate truncated predictions by using the AR representation $\pi(B)x_t = w_t$. We have

$$(1 + 0.3B - 0.1B^2)w_t = x_t,$$

which yields

$$\pi(B)(1 + 0.3B - 0.1B^2)w_t = \pi(B)x_t = w_t.$$

Hence, with $\pi(z) = 1 + \pi_1 z + \pi_2 z^2 + ...$, we need to solve

$$(1 + \pi_1 z + \pi_2 z^2 + ...)(1 + 0.3z - 0.1z^2) = 1,$$

i.e.

$$1 + (\pi_1 + 0.3)z + (\pi_2 + 0.3\pi_1 - 0.1)z^2 + ... = 1,$$

and it follows that

$$\begin{aligned}
\pi_1 &= -0.3, \\
\pi_2 &= -0.3\pi_1 + 0.1 = 0.19, \\
\pi_3 &= -0.3\pi_2 + 0.1\pi_1 = -0.087, \\
\pi_4 &= -0.3\pi_3 + 0.1\pi_2 = 0.0451, \\
\pi_5 &= -0.3\pi_4 + 0.1\pi_3 = -0.02223.
\end{aligned}$$

The truncated predictions become

$$\begin{aligned}
\tilde{x}_{201} &= -\pi_1 x_{200} - \pi_2 x_{199} - ... \\
&\approx 0.3 \cdot 0.4 - 0.19 \cdot 0.0 + 0.087 \cdot 0.4 - 0.0451 \cdot 0.8 \\
&= 0.11872
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{x}_{202} \\
&= -\pi_1 \tilde{x}_{201} - \pi_2 x_{200} - \pi_3 x_{199} - ... \\
&\approx 0.3 \cdot 0.11872 - 0.19 \cdot 0.4 + 0.087 \cdot 0.0 - 0.0451 \cdot 0.4 + 0.02223 \cdot 0.8 \\
&= -0.04064.
\end{aligned}$$

(b) Calculate 95% prediction intervals for x_{201} and x_{202}. (3p)

Solution: The mean square predicion error m steps ahead is given by $\sigma_w^2 \sum_{j=1}^{m-1} \psi_j^2$, where the ψ_j are the coefficients in the MA representation, with $\psi_0 = 1$. We only need to find ψ_1. In our case, this is easy since we have a pure MA process, hence $\psi_1 = 0.3$.

With $\sigma_w^2 = 0.16$, this gives the 95% prediction interval for x_{201} as

$$0.11872 \pm 1.96\sqrt{0.16} = 0.11872 \pm 0.784 = (-0.665,\ 0.903),$$

and for x_{202}, we find the corresponding interval

$$-0.04064 \pm 1.96\sqrt{0.16(1+0.3^2)} = -0.04064 \pm 0.81852 = (-0.859,\ 0.778).$$

5. Consider the time series model

$$x_t = 0.9x_{t-4} + w_t + 0.2w_{t-1},$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 2$.

(a) Write it as a model on the form $\text{SARMA}(p, q) \times (P, Q)_s$. (2p)

Solution: The general form of a $\text{SARMA}(p, q) \times (P, Q)_s$ model is

$$\Phi_P(B^s)\phi_p(B)x_t = \Theta_Q(B^s)\theta(B)w_t.$$

We may write our model as

$$(1 - 0.9B^4)x_t = (1 + 0.2B)w_t,$$

i.e. $s = 4$, $\Phi_P(B^s) = 1 - 0.9B^4$, $\phi_p(B) = \Theta_Q(B^s) = 1$ and $\theta(B) = 1 + 0.2B$. Hence, we have a $\text{SARMA}(p, q) \times (P, Q)_s$ model with $p = 0$, $q = 1$, $P = 1$, $Q = 0$ and $s = 4$.

(b) Calculate the spectral density of x_t at the frequency $\omega = 0.25$. (2p)

Solution: We may use the general formula

$$f(\omega) = \sigma_w^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2}.$$

In our case, we have $\sigma_w^2 = 2$, and using $|z|^2 = z\bar{z}$ where $\bar{z}$ is the complex conjugate of z,

$$\begin{aligned}|\theta(e^{-2\pi i\omega})|^2 &= |1 + 0.2e^{-2\pi i\omega}|^2 = (1 + 0.2e^{-2\pi i\omega})(1 + 0.2e^{2\pi i\omega}) \\ &= 1.04 + 0.4\frac{e^{2\pi i\omega} + e^{-2\pi i\omega}}{2} = 1.04 + 0.4\cos(2\pi\omega),\end{aligned}$$

and

$$\begin{aligned}|\phi(e^{-2\pi i\omega})|^2 &= |1 - 0.9(e^{-2\pi i\omega})^4|^2 = (1 - 0.9e^{-8\pi i\omega})(1 - 0.9e^{8\pi i\omega}) \\ &= 1.81 - 1.8\frac{e^{8\pi i\omega} + e^{-8\pi i\omega}}{2} = 1.81 - 1.8\cos(8\pi\omega).\end{aligned}$$

Hence,

$$f(\omega) = 2 \cdot \frac{1.04 + 0.4\cos(2\pi\omega)}{1.81 - 1.8\cos(8\pi\omega)},$$

and inserting $\omega = 0.25 = 1/4$, we get

$$f(0.25) = 2 \cdot \frac{1.04 + 0.4\cos(\pi/2)}{1.81 - 1.8\cos(2\pi)} = 2 \cdot \frac{1.04}{0.01} = 208.$$

(c) Calculate the spectral density of $y_t = x_t - x_{t-4}$ at the frequency $\omega = 0.25$. (2p)

Solution: We have $y_t = a(B)x_t$ where $a(B) = \sum_j a_j B^j$ with $a_0 = 1$, $a_4 = -1$ and $a_j = 0$ otherwise, which yields the frequency response function

$$A_{yx}(\omega) = \sum_j a_j e^{-2\pi i \omega j} = 1 - e^{-8\pi i \omega}.$$

Thus, in the same manner as above,

$$|A_{yx}(\omega)|^2 = |1 - e^{-8\pi i \omega}|^2 = (1 - e^{-8\pi i \omega})(1 - e^{8\pi i \omega}) = 2 - 2\cos(8\pi\omega),$$

and so,

$$|A_{yx}(0.25)|^2 = 2 - 2\cos(2\pi) = 0.$$

This means that the spectral density of y at $\omega = 0.25$ is

$$f_{yy}(0.25) = |A_{yx}(0.25)|^2 f_{xx}(0.25) = 0.$$

(d) Compare and discuss your results in (b) and (c). (1p)

Solution: It is natural that the spectral density in (b) is high at $\omega = 0.25 = 1/4$, since we have a seasonal model with season 4.

In (c), we use a seasonal difference, which kills the spectral density at the corresponding seasonal frequency 1/4, hence it becomes zero.

6. Three data series were collected from the website of Statistics Sweden (SCB): The monthly number of people in the workforce 2001-2024 (Figure 1), the quarterly electrical energy balance in TJ 1984-2017 (Figure 2) and the population size 1880-2023 (Figure 3).

   In Figures 4-7, estimated spectral densities of the series of Figures 1-3 are given in 'random' order, together with an estimated spectral density from another series (not shown).

   Match figures 1-3 with three of the figures 4-7. Motivate your answer. (5p)

   Solution: The series in figure 1 has both a clear trend and a season of 12 (being a monthly series, and we can also see from the plot that there is one peak each year). Hence, it should correspond to a spectral density that is high close to zero (trend) and at multiples of $1/12 \approx 0.08$. This is what we see in figure 6.

   The series in figure 2 is a quarterly series, hence it has a season of 4. It also has some kind of trending behaviour. Thus, the spectral density may be high close to zero and at the frequencies $1/4 = 0.25$ and $2 \cdot 0.25 = 0.5$. This is what we find in figure 4.

   Finally, figure 3 depicts a series with a pronounced trend but not much more seasonal structure. Hence, the corresponding spectral density should be high close to zero and have no other peaks. We find this in figure 7.

   The spectral density in figure 5 does not seem to match any of the series in figures 1-3.

   To sum up, the matching should be 1-6, 2-4, 3-7.

7. Consider the system

$$\begin{aligned} x_{1,t} &= w_{1,t}, \\ x_{2,t} &= -0.4x_{1,t-1} + 0.5w_{1,t-1} + w_{2,t}, \end{aligned}$$

where $\{w_{1,t}\}$ and $\{w_{2,t}\}$ are white noise processes (possibly dependent of each other).

(a) Write this system on vector/matrix form as a VARMA(p, q) model. What is p and q here? (2p)

Solution: On matrix form, the system is

$$(I - \Phi B)\mathbf{x}_t = (I + \Theta B)\mathbf{w}_t,$$

i.e. $p = 1$, $q = 1$, where I is a two-dimensional identity matrix, $\mathbf{x}_t = (x_{1,t}, x_{2,t})'$, $\mathbf{w}_t = (w_{1,t}, w_{2,t})'$ and

$$\Phi = \begin{pmatrix} 0 & 0 \\ -0.4 & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \end{pmatrix}.$$

(b) Show that this model is equivalent to a VARMA$(0, 1)$ model. (3p)

Solution: Because

$$(I - \Phi B)^{-1} = \begin{pmatrix} 1 & 0 \\ 0.4B & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -0.4B & 1 \end{pmatrix},$$

it follows that

$$\mathbf{x}_t = \begin{pmatrix} 1 & 0 \\ -0.4B & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0.5B & 1 \end{pmatrix} \mathbf{w}_t = \begin{pmatrix} 1 & 0 \\ 0.1B & 1 \end{pmatrix} \mathbf{w}_t,$$

which is a VARMA$(0, 1)$ model $\mathbf{x}_t = (I + \Theta_1 B)\mathbf{w}_t$ with

$$\Theta_1 = \begin{pmatrix} 0 & 0 \\ 0.1 & 0 \end{pmatrix}.$$

Eqvivalently (maybe simpler), $x_{1,t-1} = w_{1,t-1}$ may be inserted into the equation for $x_{2,t}$ to yield

$$x_{2,t} = -0.4w_{1,t-1} + 0.5w_{1,t-1} + w_{2,t} = 0.1w_{1,t-1} + w_{2,t}.$$

Together with $x_{1,t} = w_{1,t}$, this gives the system as above.

Appendix: figures

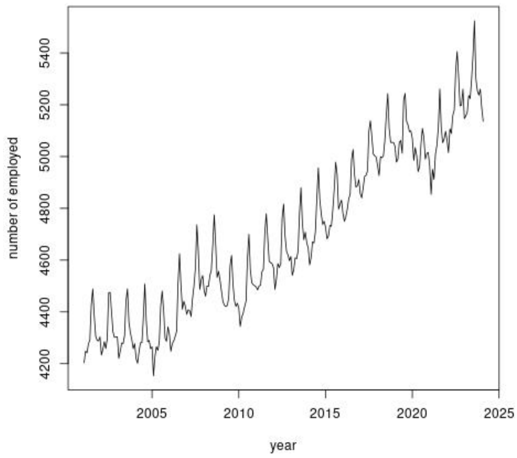

Figure 1: The monthly number of people in the workforce.

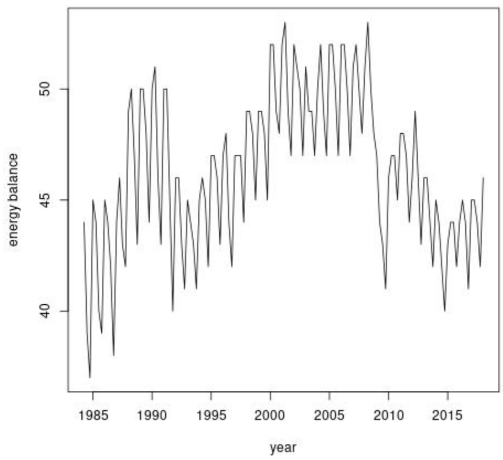

Figure 2: The quarterly electrical energy balance.

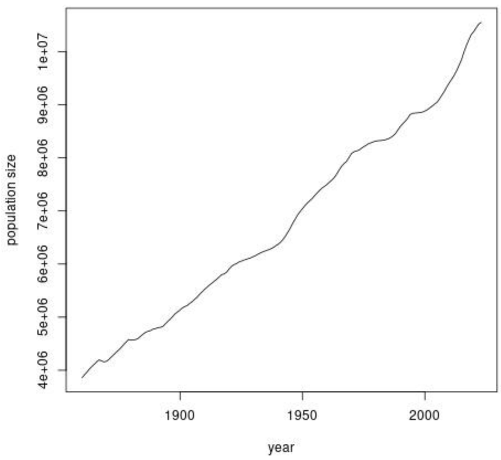

Figure 3: The population size.

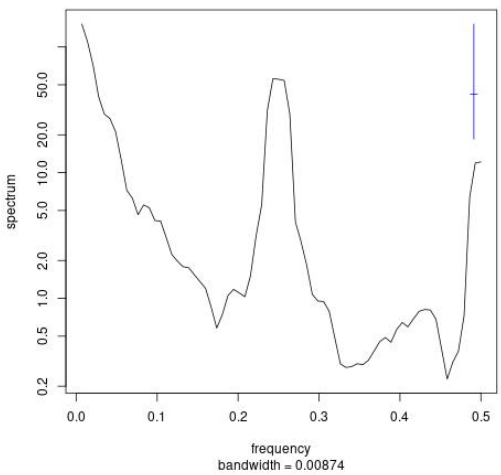

Figure 4: Estimated spectral density, problem 6.

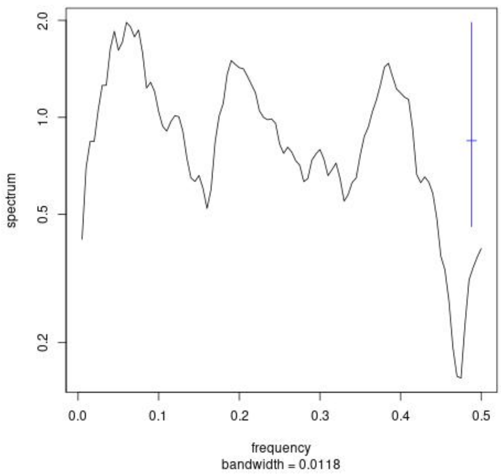

Figure 5: Estimated spectral density, problem 6.

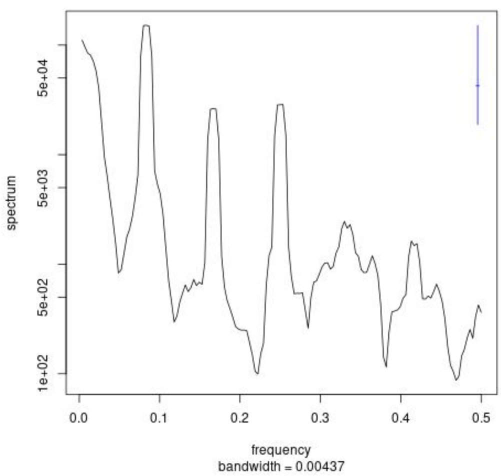

Figure 6: Estimated spectral density, problem 6.

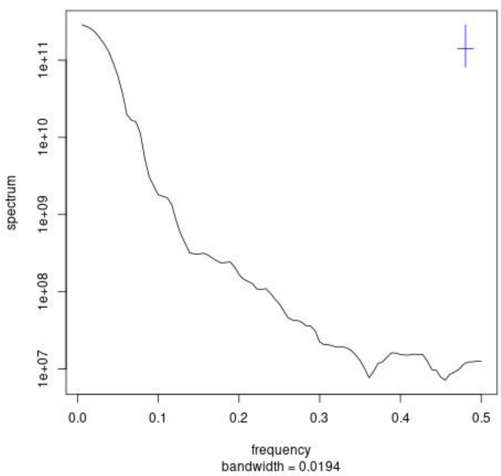

Figure 7: Estimated spectral density, problem 6.