UPPSALA UNIVERSITY
Department of Mathematics
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Exam in Mathematical Statistics
Analysis of Time Series 1MS014
2025–05–26

Time: 8.00-13.00. Limits for the credits 3, 4, 5 are 18, 25 and 32 points, respectively, including bonus points. The solutions should be well motivated.

Permitted aids: The course book or copies thereof. Hand-written sheet of formulae. Pocket calculator. Dictionary. *No electronic device with internet connection.*

1. Let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 1$ and let
$$x_t = 2 + 0.6w_t^2 + 0.2w_{t-1}^2.$$
Calculate the mean and autocovariance function of x_t and state whether it is weakly stationary. (5p)

   Hint: If Z is a standard normal random variable, then $E(Z^4) = 3$.

2. For the ARMA(p, q) models below, where $\{w_t\}$ are white noise processes, find p and q and determine whether they are causal and/or invertible. (6p)

   (a) $x_t = 0.8x_{t-1} + w_t + 0.8w_{t-1}$

   (b) $x_t = 0.8x_{t-1} + w_t + 0.64w_{t-2}$

   (c) $x_t = 0.5x_{t-1} + 0.5x_{t-2} + w_t$

   (d) $x_t = 0.7x_{t-1} - 0.1x_{t-2} + w_t - 0.5w_{t-1}$

3. Let $\{w_t\}$ be a white noise process with variance $\sigma_w^2 = 1$ and define x_t through
$$x_t = 0.5x_{t-1} + w_t + 0.5w_{t-2}.$$
Calculate the autocorrelation function $\rho(h)$ for $h = 1, 2, 3, 4$. (6p)

   Hint: You may assume that $\text{cov}(x_{t-1}, w_{t-2}) = \text{cov}(x_t, w_{t-1})$ and $\text{cov}(x_{t-1}, w_{t-1}) = \text{cov}(x_t, w_t)$.

4. Consider the process
$$x_t = 0.6x_{t-4} + w_t - 0.8w_{t-1}$$
where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 0.2$. We observe x_t up to time $t = 300$, where the last four observations are $x_{297} = 1.0$, $x_{298} = 0.4$, $x_{299} = 0.2$ and $x_{300} = 0.1$.

   (a) Predict the values of x_{301} and x_{302}. Approximations are permitted. (4p)

   (b) Calculate 95% prediction intervals for x_{301} and x_{302}. (2p)

5. As in problem 1, let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 1$ and let
$$x_t = 2 + 0.6w_t^2 + 0.2w_{t-1}^2.$$
   (a) Calculate the spectral density of x_t. (2p)
   (b) Calculate the spectral density of $y_t = x_t - x_{t-1}$. (2p)
   (c) What is the value of the spectral density $f_y(\omega)$ of y_t at $\omega = 0$? Interpret this result. (1p)

6. The time series x_t gives the total electricity supply in Sweden in GWh as a monthly series starting 1974 and ending February 2025. The series is plotted in Figure 1. Let
$$y_t = x_t - x_{t-1},$$
$$z_t = \frac{1}{12}\sum_{j=1}^{12} x_{t+1-j}.$$
   In Figures 2-5, the estimated spectral densities (non parametric in R, `spans`= 8) are plotted for x_t, y_t, z_t and an unrelated series, in 'random' order. Match x_t, y_t, and z_t with one figure each. (5p)

7. Consider the ARCH model
$$y_t = \sigma_t \epsilon_t,$$
$$\sigma_t^2 = 1 + \frac{1}{4}y_{t-1}^2,$$
   where the ϵ_t are i.i.d. $N(0, 1)$ (standard normal).
   (a) Calculate $E(y_t)$. (1p)
   (b) Calculate $\text{Var}(y_t)$. (2p)
   (c) Calculate $E(y_t^6)$. (4p)

   Without proof, you may assume that y_t is stationary, and that if Z is $N(0, 1)$, then $E(Z^4) = 3$ and $E(Z^6) = 15$.

GOOD LUCK!

Appendix: figures

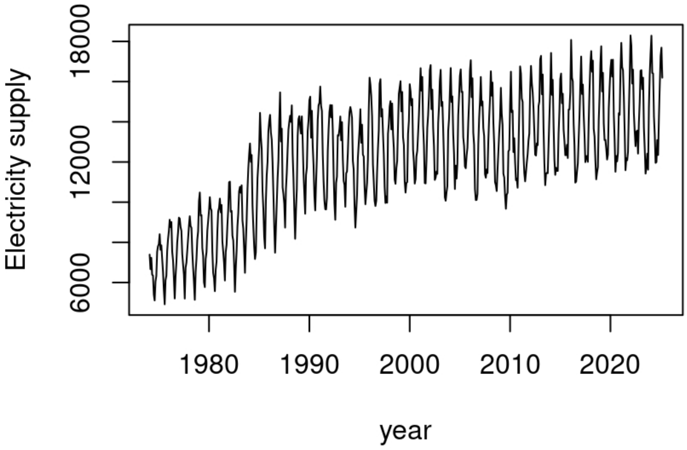

Figure 1: The electricity supply in Sweden.

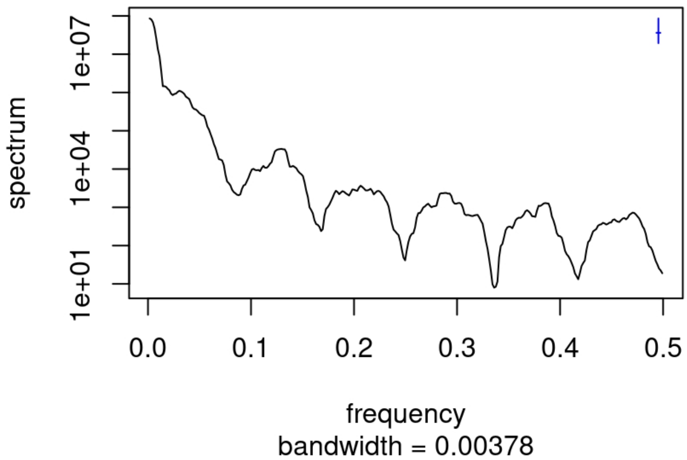

Figure 2: Estimated spectral density, problem 6.

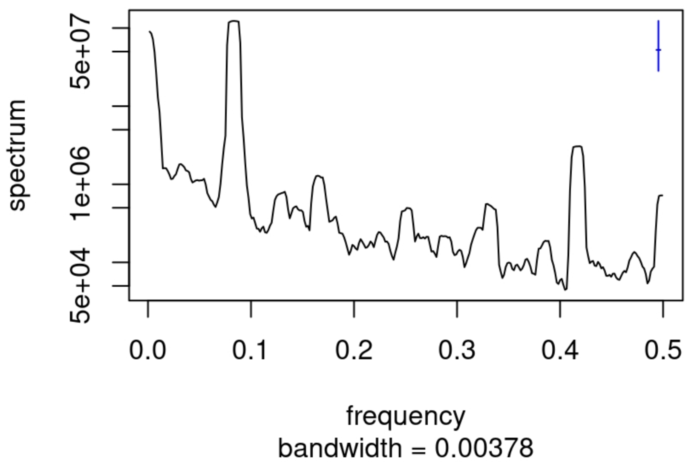

Figure 3: Estimated spectral density, problem 6.

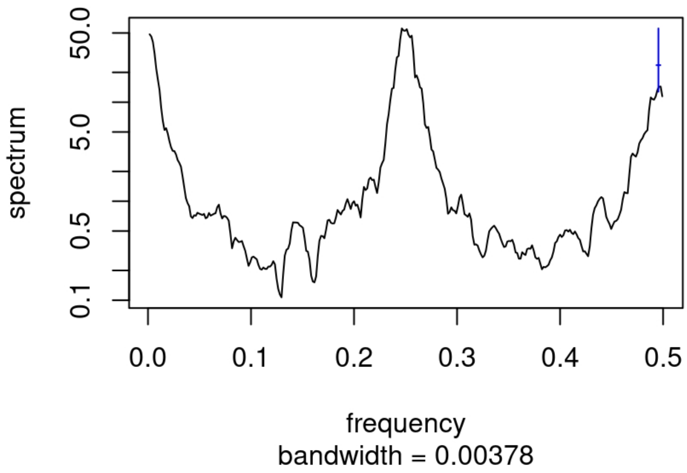

Figure 4: Estimated spectral density, problem 6.

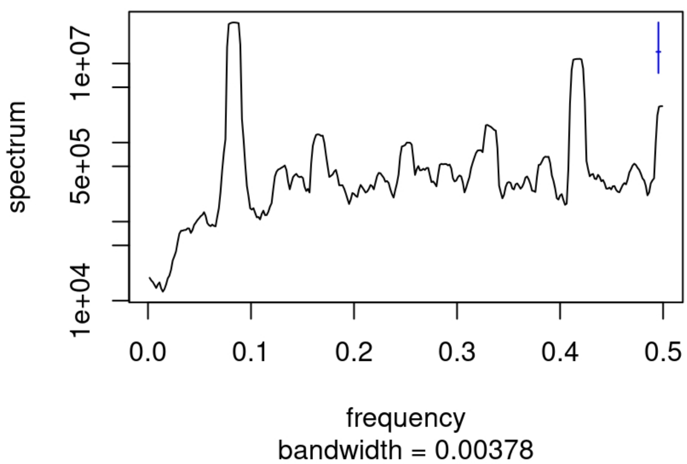

Figure 5: Estimated spectral density, problem 6.