UPPSALA UNIVERSITY
Department of Mathematics
Rolf Larsson

Exam in Mathematical Statistics
Analysis of Time Series 1MS014
2025–05–26
Solutions

1. Let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 1$ and let
$$x_t = 2 + 0.6w_t^2 + 0.2w_{t-1}^2.$$
Calculate the mean and autocovariance function of x_t and state whether it is weakly stationary. (5p)

Hint: If Z is a standard normal random variable, then $E(Z^4) = 3$.

Solution: The mean function is given by
$$\mu_t = E(x_t) = 2 + 0.6E(w_t^2) + 0.2E(w_{t-1}^2) = 2 + 0.6 \cdot 1 + 0.2 \cdot 1 = 2.8.$$
As for the autocovariance function, since the covariance between a constant and a random variable is always zero, and because for all t,
$$\text{cov}(w_t^2, w_t^2) = E(w_t^4) - \{E(w_t^2)\}^2 = 3\sigma_w^4 - \sigma_w^4 = 2\sigma_w^4 = 2 \cdot 1^2 = 2,$$
while by independence, $\text{cov}(w_{t+h}^2, w_t^2) = 0$ if $h \neq 0$, we have
$$\begin{aligned}
&\gamma(t+h, t) \\
&= \text{cov}(x_{t+h}, x_t) = \text{cov}(2 + 0.6w_{t+h}^2 + 0.2w_{t+h-1}^2, 2 + 0.6w_t^2 + 0.2w_{t-1}^2) \\
&= 0.6^2\text{cov}(w_{t+h}^2, w_t^2) + 0.6 \cdot 0.2\text{cov}(w_{t+h}^2, w_{t-1}^2) \\
&+ 0.2 \cdot 0.6\text{cov}(w_{t+h-1}^2, w_t^2) + 0.2^2\text{cov}(w_{t+h-1}^2, w_{t-1}^2) \\
&= 0.36 \cdot 2I\{h = 0\} + 0.12 \cdot 2I\{h = -1\} + 0.12 \cdot 2I\{h = 1\} \\
&+ 0.04 \cdot 2I\{h = 0\} \\
&= 0.8I\{h = 0\} + 0.24I\{|h| = 1\} \\
&= \begin{cases} 0.8 & \text{if } h = 0, \\ 0.24 & \text{if } |h| = 1, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$
where $I\{A\} = 1$ if A is fulfilled and 0 otherwise.

Because μ_t is constant, $\text{var}x_t = \gamma(t, t) = 0.8$ is constant and finite and since $\gamma(t+h, t)$ is not a function of t, x_t is weakly stationary.

2. For the ARMA(p, q) models below, where $\{w_t\}$ are white noise processes, find p and q and determine whether they are causal and/or invertible. (6p)

   (a) $x_t = 0.8x_{t-1} + w_t + 0.8w_{t-1}$

   Solution: This model can be described as $\phi(B)x_t = \theta(B)w_t$, where $\phi(B) = 1 - 0.8B$ and $\theta(B) = 1 + 0.8B$. No simplification of the model is possible (no common roots), hence this is an ARMA(1,1) model, i.e. $p = 1$ and $q = 1$.

   To check causality, we find that $0 = \phi(z) = 1 - 0.8z$ has the solution $z = 1/0.8 = 1.25 > 1$, hence the model is causal. It is also invertible, because $0 = \theta(z) = 1 + 0.8z$ is solved by $z = -1/0.8 = -1.25$, and $|-1.25| = 1.25 > 1$.

   (b) $x_t = 0.8x_{t-1} + w_t + 0.64w_{t-2}$

   Solution: Here, $\phi(B)x_t = \theta(B)w_t$ with $\phi(B) = 1 - 0.8B$ and $\theta(B) = 1+0.64B^2$. The solution of $0 = \phi(z) = 1-0.8z$ is $z = 1.25 > 1$ as in (a), and this is not a root for $\theta(z)$, hence this is a ARMA(1,2) model, or alternatively SARMA$(1, 0) \times (0, 1)_2$.

   As we saw above, the root of $\phi(z)$ is outside the complex unit circle, hence the model is causal. As for invertibility, we have $0 = \theta(z) = 1 + 0.64z^2$, which implies $z^2 = -1/0.64$, i.e. $z = \pm i/0.8 = \pm 1.25i$, where i is the complex unit $i = \sqrt{-1}$. Since $|\pm 1.25i|^2 = 1.25^2 > 1$, the model is invertible.

   (c) $x_t = 0.5x_{t-1} + 0.5x_{t-2} + w_t$

   Solution: This is clearly a AR(2) model $\phi(B)x_t = w_t$ with $\phi(B) = 1 - 0.5B - 0.5B^2$.

   It is invertible, since all AR models are. Is it causal? The equation $0 = \phi(z) = 1 - 0.5z - 0.5z^2$ is equivalent to $z^2 + z - 2 = 0$, with solutions

$$z_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2} = \frac{-1 \pm 3}{2},$$

   i.e. $z_1 = 1$ and $z_2 = -2$. Since $|z_1| = 1$, this root is on (and not outside) the complex unit circle. Hence, the model is *not* causal.

(d) $x_t = 0.7x_{t-1} - 0.1x_{t-2} + w_t - 0.5w_{t-1}$

Solution: This appears to be a ARMA(2,1) model $\phi(B)x_t = \theta(B)w_t$ with $\phi(B) = 1 - 0.7B + 0.1B^2$ and $\theta(B) = 1 - 0.5B$, but we need to check if $\phi(z)$ and $\theta(z)$ have any common roots. Here $0 = \theta(z) = 1 - 0.5z$ for $z = 2$, and in fact, $\phi(2) = 1 - 0.7 \cdot 2 + 0.2 \cdot 2^2 = 0$, so there is a common root. Solving $0 = \phi(z) = 1 - 0.7z + 0.1z^2$ is equivalent to $z^2 - 7z + 10 = 0$, which has the solutions

$$z_{1,2} = \frac{7}{2} \pm \sqrt{\frac{49}{4} - 10} = \frac{7 \pm 3}{2},$$

giving $z_1 = 5$ and $z_2 = 2$. By the factorization theorem, this means that

$$\phi(z) = 0.1(z - 5)(z - 2) = (1 - 0.2z)(1 - 0.5z),$$

so the model may be written as

$$(1 - 0.2B)(1 - 0.5B)x_t = (1 - 0.5B)w_t,$$

i.e. $(1 - 0.2B)x_t = w_t$, an AR(1) model with $\phi(B) = 1 - 0.2B$.
The model is invertible, since all AR models are. It is also causal, since $0 = 1 - 0.2z$ gives $z = 5 > 1$.

3. Let $\{w_t\}$ be a white noise process with variance $\sigma_w^2 = 1$ and define x_t through

$$x_t = 0.5x_{t-1} + w_t + 0.5w_{t-2}.$$

Calculate the autocorrelation function $\rho(h)$ for $h = 1, 2, 3, 4$. (6p)

Hint: You may assume that $\text{cov}(x_{t-1}, w_{t-2}) = \text{cov}(x_t, w_{t-1})$ and $\text{cov}(x_{t-1}, w_{t-1}) = \text{cov}(x_t, w_t)$.

Solution: At first, we derive the autocorrelation function.

Observe that $\text{cov}(x_{t-h}, w_t) = 0$ for all $h \geq 1$. From the model equation, we have

$$\begin{aligned}\gamma(0) &= \text{cov}(x_t, x_t) = \text{cov}(0.5x_{t-1} + w_t + 0.5w_{t-2}, 0.5x_{t-1} + w_t + 0.5w_{t-2}) \\ &= \frac{1}{4}\text{cov}(x_{t-1}, x_{t-1}) + \text{cov}(w_t, w_t) + \frac{1}{4}\text{cov}(w_{t-2}, w_{t-2}) + \frac{1}{2}\text{cov}(x_{t-1}, w_{t-2}) \\ &= \frac{1}{4}\gamma(0) + \frac{5}{4} + \frac{1}{2}\text{cov}(x_t, w_{t-1}),\end{aligned}$$

by the hint. Hence, solving for $\gamma(0)$ we obtain

$$\gamma(0) = \frac{5}{3} + \frac{2}{3}\text{cov}(x_t, w_{t-1}). \tag{1}$$

Similarly, we find

$$\begin{aligned}\gamma(1) &= \text{cov}(x_{t+1}, x_t) = \text{cov}(0.5x_t + w_{t+1} + 0.5w_{t-1}, x_t) \\ &= \frac{1}{2}\gamma(0) + \frac{1}{2}\text{cov}(x_t, w_{t-1}),\end{aligned} \tag{2}$$

$$\begin{aligned}\gamma(2) &= \text{cov}(x_{t+2}, x_t) = \text{cov}(0.5x_{t+1} + w_{t+2} + 0.5w_t, x_t) \\ &= \frac{1}{2}\gamma(1) + \frac{1}{2}\text{cov}(x_t, w_t),\end{aligned} \tag{3}$$

and for $h \geq 3$,

$$\begin{aligned}\gamma(h) &= \text{cov}(x_{t+h}, x_t) = \text{cov}(0.5x_{t+h-1} + w_{t+h} + 0.5w_{t+h-1}, x_t) \\ &= \frac{1}{2}\gamma(h-1).\end{aligned} \tag{4}$$

Moreover,

$$\text{cov}(x_t, w_t) = \text{cov}(0.5x_{t-1} + w_t + 0.5w_{t-2}, w_t) = \text{cov}(w_t, w_t) = 1,$$

and

$$\begin{aligned}\text{cov}(x_t, w_{t-1}) &= \text{cov}(0.5x_{t-1} + w_t + 0.5w_{t-2}, w_{t-1}) = \frac{1}{2}\text{cov}(x_{t-1}, w_{t-1}) \\ &= \frac{1}{2}\text{cov}(x_t, w_t) = \frac{1}{2},\end{aligned}$$

by the hint.

Hence, by (1)-(3),

$$\gamma(0) = \frac{5}{3} + \frac{2}{3} \cdot \frac{1}{2} = 2,$$
$$\gamma(1) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{4},$$
$$\gamma(2) = \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot 1 = \frac{9}{8}.$$

This gives

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{5}{8} = 0.625,$$
$$\rho(2) = \frac{\gamma(2)}{\gamma(0)} = \frac{9}{16} = 0.5625.$$

Moreover, from (4),

$$\rho(3) = \frac{\gamma(3)}{\gamma(0)} = \frac{1}{2}\frac{\gamma(2)}{\gamma(0)} = \frac{1}{2}\rho(2) = \frac{9}{32} = 0.28125,$$

and similarly,

$$\rho(4) = \frac{1}{2}\rho(3) = \frac{9}{64} = 0.140625.$$

4. Consider the process

$$x_t = 0.6x_{t-4} + w_t - 0.8w_{t-1}$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 0.2$. We observe x_t up to time $t = 300$, where the last four observations are $x_{297} = 1.0$, $x_{298} = 0.4$, $x_{299} = 0.2$ and $x_{300} = 0.1$.

(a) Predict the values of x_{301} and x_{302}. Approximations are permitted. (4p)

Solution: We will calculate truncated predictions by using the AR representation $\pi(B)x_t = w_t$. We have

$$(1 - 0.8B)w_t = (1 - 0.6B^4)x_t,$$

which yields

$$\pi(B)(1 - 0.8B)w_t = (1 - 0.6B^4)\pi(B)x_t = (1 - 0.6B^4)w_t.$$

Hence, with $\pi(z) = 1 + \pi_1 z + \pi_2 z^2 + ...$, we need to solve

$$(1 + \pi_1 z + \pi_2 z^2 + \pi_3 z^3 + ...)(1 - 0.8z) = 1 - 0.6z^4,$$

i.e.

$$1 + (\pi_1 - 0.8)z + (\pi_2 - 0.8\pi_1)z^2 + (\pi_3 - 0.8\pi_2)z^3 + (\pi_4 - 0.8\pi_3)z^4 \\ + (\pi_5 - 0.8\pi_4)z^5 + ... = 1 - 0.6z^4,$$

which yields

$$\begin{aligned} \pi_1 &= 0.8, \\ \pi_2 &= 0.8\pi_1 = 0.8^2 = 0.64, \\ \pi_3 &= 0.8^3 = 0.512, \\ \pi_4 &= 0.8^4 - 0.6 = -0.1904, \\ \pi_5 &= 0.8^5 - 0.6 \cdot 0.8 = -0.15232. \end{aligned}$$

The truncated predictions become

$$\begin{aligned} \tilde{x}_{301} &= -\pi_1 x_{300} - \pi_2 x_{299} - ... \\ &\approx -0.8 \cdot 0.1 - 0.64 \cdot 0.2 - 0.512 \cdot 0.4 + 0.1904 \cdot 1.0 = -0.2224, \end{aligned}$$

and

$$\begin{aligned} \tilde{x}_{302} &= -\pi_1 \tilde{x}_{301} - \pi_2 x_{300} - \pi_3 x_{299} - ... \\ &\approx -0.8 \cdot (-0.2224) - 0.64 \cdot 0.1 - 0.512 \cdot 0.2 \\ &\quad + 0.1904 \cdot 0.4 + 0.15232 \cdot 1.0 \\ &= 0.24. \end{aligned}$$

(b) Calculate 95% prediction intervals for x_{301} and x_{302}. (2p)

Solution: The mean square predicion error m steps ahead is given by $\sigma_w^2 \sum_{j=1}^{m-1} \psi_j^2$, where the ψ_j are the coefficients in the MA representation, with $\psi_0 = 1$. We only need to find ψ_1. To this end, $x_t = \psi(B)w_t$ implies

$$\psi(B)(1 - 0.6B^4)x_t = (1 - 0.8B)\psi(B)w_t = (1 - 0.8B)x_t,$$

so that with $\psi(z) = 1 + \psi_1 z + ...$, we have

$$(1 + \psi_1 z + ...)(1 - 0.6z^4) = 1 - 0.8z,$$

implying $\psi_1 = -0.8$.

With $\sigma_w^2 = 0.2$, this gives the 95% prediction interval for x_{301} as

$$-0.2224 \pm 1.96\sqrt{0.2} \approx -0.2224 \pm 0.8765 \approx (-1.10,\ 0.65),$$

and for x_{302}, we find the corresponding interval

$$0.24 \pm 1.96\sqrt{0.1\{1 + (-0.8)^2\}} \approx 0.24 \pm 1.12 \approx (-0.88,\ 1.36).$$

5. As in problem 1, let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 1$ and let

$$x_t = 2 + 0.6w_t^2 + 0.2w_{t-1}^2.$$

(a) Calculate the spectral density of x_t. (2p)

Solution: In problem 1, we got $\gamma(0) = 0.8$, $\gamma(\pm 1) = 0.24$ and $\gamma(h) = 0$ for all other h. The spectral density is the Fourier transform of $\gamma(h)$, i.e.

$$\begin{aligned} f(\omega) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h} = 0.8 \cdot e^0 + 0.24 \left(e^{2\pi i \omega} + e^{-2\pi i \omega} \right) \\ &= 0.8 + 0.48 \cos(2\pi\omega). \end{aligned}$$

(b) Calculate the spectral density of $y_t = x_t - x_{t-1}$. (2p)

Solution: We have $y_t = A_{yx}(B)x_t$, where $A_{yx}(B) = \sum_j a_j B^j$ with $a_0 = 1$, $a_1 = -1$ and $a_j = 0$ otherwise. We may use the result

$$f_{yy}(\omega) = |A_{yx}(\omega)|^2 f_{xx}(\omega),$$

where we know $f_{xx}(\omega)$ from (a) and

$$A_{yx}(\omega) = \sum_j a_j e^{-2\pi i \omega j} = 1 - e^{-2\pi i \omega},$$

which gives

$$\begin{aligned} |A_{yx}(\omega)|^2 &= A_{yx}(\omega)\overline{A_{yx}(\omega)} = \left(1 - e^{-2\pi i \omega}\right)\left(1 - e^{2\pi i \omega}\right) \\ &= 2 - e^{2\pi i \omega} - e^{-2\pi i \omega} = 2\left\{1 - \cos(2\pi\omega)\right\}. \end{aligned}$$

Hence, the spectral density for y_t is

$$\begin{aligned} f_{yy}(\omega) &= 2\left\{1 - \cos(2\pi\omega)\right\}\left\{0.8 + 0.48\cos(2\pi\omega)\right\} \\ &= 1.6 - 0.64\cos(2\pi\omega) - 0.96\cos^2(2\pi\omega). \end{aligned}$$

(c) What is the value of the spectral density $f_y(\omega)$ of y_t at $\omega = 0$? Interpret this result. (1p)

Solution: Because $\cos(2\pi \cdot 0) = \cos(0) = 1$, we easily find from (b) that $f_{yy}(0) = 0$. This is natural, since differencing kills the trend, amd the trend corresponds to frequency 0.

6. The time series x_t gives the total electricity supply in Sweden in GWh as a monthly series starting 1974 and ending February 2025. The series is plotted in Figure 1. Let

$$y_t = x_t - x_{t-1},$$
$$z_t = \frac{1}{12}\sum_{j=1}^{12} x_{t+1-j}.$$

In Figures 2-5, the estimated spectral densities (non parametric in R, `spans`= 8) are plotted for x_t, y_t, z_t and an unrelated series, in 'random' order. Match x_t, y_t, and z_t with one figure each. (5p)

Solution: From the plot of the series in Figure 1, we can see that the series exhibits a trend and yearly seasonality, and since the series is monthly, this means a season of length 12, i.e. a frequency of $1/12 \approx 0.08$. This would correspond to a spectral density with a peak near zero (corresponding to the trend) and peaks at multiples of about 0.08, corresponding to season. This is what we find in Figure 3.

When differencing the series as for y_t, the trend cancels out and the peak at zero in the spectral density disappears, while the seasonal peaks stay. This is what we see in Figure 5.

Finally, z_t is the series obtained after taking a yearly moving average. This should take away the seasonal peaks but not the trend peak at zero. We have this in Figure 2.

(The series in Figure 4 exhibits a trend peak and a peak at around 0.25, which corresponds to a season length of 4. None of our three alternatives corresponds to that.)

7. Consider the ARCH model

$$y_t = \sigma_t \epsilon_t,$$
$$\sigma_t^2 = 1 + \frac{1}{4} y_{t-1}^2,$$

where the ϵ_t are i.i.d. $N(0,1)$ (standard normal).

(a) Calculate $E(y_t)$. (1p)

Solution: Let $Y_s = \{y_s, y_{s-1}, ...\}$. i.e. all information gathered up to time s. By the law of iterated expectations, we have

$$E(y_t) = E\{E(y_t|Y_{t-1})\} = E\{E(\sigma_t \epsilon_t|Y_{t-1})\} = E\{\sigma_t E(\epsilon_t|Y_{t-1})\}$$
$$= E\{\sigma_t E(\epsilon_t)\} = 0.$$

Here, the third equality follows because σ_t is a function of Y_{t-1}, the fourth equality follows because ϵ_t is independent of Y_{t-1}, and the last equality follows since $E(\epsilon_t) = 0$.

(b) Calculate $\mathrm{Var}(y_t)$. (2p)

Solution: Because $y_t^2 = \sigma_t^2 \epsilon_t^2$, we get

$$y_t^2 - \left(1 + \frac{1}{4} y_{t-1}^2\right) = \sigma_t^2 \epsilon_t^2 - \sigma_t^2 = v_t,$$

where $v_t = \sigma_t^2(\epsilon_t^2 - 1)$, i.e.

$$y_t^2 = 1 + \frac{1}{4} y_{t-1}^2 + v_t. \quad (5)$$

As in (a), we have that

$$E(v_t) = E\{E(v_t|Y_{t-1}\} = E\{\sigma_t^2 E(\epsilon_t^2 - 1|Y_{t-1})\} = E\{\sigma_t^2 E(\epsilon_t^2 - 1)\}$$
$$= E[\sigma_t^2\{E(\epsilon_t^2) - 1\}] = E[\sigma_t^2(1-1)] = 0.$$

Hence, because $E(y_t) = 0$, we have from (5) that

$$\mathrm{Var}(y_t) = E(y_t^2) = 1 + \frac{1}{4} E(y_{t-1}^2) + E(v_t) = 1 + \frac{1}{4}\mathrm{Var}(y_{t-1}). \quad (6)$$

Because of stationarity, $\mathrm{Var}(y_{t-1}) = \mathrm{Var}(y_t)$, and we may solve for $\mathrm{Var}(y_t)$ to obtain

$$\mathrm{Var}(y_t) = \frac{4}{3} \approx 1.33.$$

Alternatively (and simpler), we may note that

$$E(y_t^2|Y_{t-1}) = E(\sigma_t^2 \epsilon_t^2|Y_{t-1}) = \sigma_t^2 E(\epsilon_t^2) = \sigma_t^2 = 1 + \frac{1}{4} y_{t-1}^2,$$

and taking expectations on both sides, we obtain equation (6).

(c) Calculate $E(y_t^6)$. (4p)

Without proof, you may assume that y_t is stationary, and that if Z is $N(0,1)$, then $E(Z^4) = 3$ and $E(Z^6) = 15$.

Solution: We will use

$$E(y_t^6) = E\{E(y_t^6|Y_{t-1})\},$$

where, similar to (a),

$$\begin{aligned} E(y_t^6|Y_{t-1}) &= E(\sigma_t^6\epsilon_t^6|Y_{t-1}) = \sigma_t^6 E(\epsilon_t^6|Y_{t-1}) = \sigma_t^6 E(\epsilon_t^6) = 15\sigma_t^6 \\ &= 15\left(1 + \frac{1}{4}y_{t-1}^2\right)^3 = 15 + \frac{45}{4}y_{t-1}^2 + \frac{45}{16}y_{t-1}^4 + \frac{15}{64}y_{t-1}^6. \end{aligned} \quad (7)$$

To find the expectation of y_{t-1}^4, we similarly get

$$\begin{aligned} E(y_t^4|Y_{t-1}) &= E(\sigma_t^4\epsilon_t^4|Y_{t-1}) = \sigma_t^4 E(\epsilon_t^4|Y_{t-1}) = \sigma_t^4 E(\epsilon_t^4) = 3\sigma_t^4 \\ &= 3\left(1 + \frac{1}{4}y_{t-1}^2\right)^2 = 3 + \frac{3}{2}y_{t-1}^2 + \frac{3}{16}y_{t-1}^4, \end{aligned}$$

and taking expectation, using stationarity and (b), we find

$$E(y_t^4) = 3 + \frac{3}{2}E(y_t^2) + \frac{3}{16}E(y_t^4) = 3 + \frac{3}{2}\cdot\frac{4}{3} + \frac{3}{16}E(y_t^4) = 5 + \frac{3}{16}E(y_t^4),$$

implying

$$E(y_t^4) = \frac{16}{13}\cdot 5 = \frac{80}{13}.$$

Thus, taking expectations in (7), we get from stationarity that

$$E(y_t^6) = 15 + \frac{45}{4}\cdot\frac{4}{3} + \frac{45}{16}\cdot\frac{80}{13} + \frac{15}{64}E(y_t^6) = 30 + \frac{225}{13} + \frac{15}{64}E(y_t^6),$$

and solving for $E(y_t^6)$ yields

$$E(y_t^6) = \frac{64}{49}\left(30 + \frac{225}{13}\right) \approx 61.79.$$

Appendix: figures

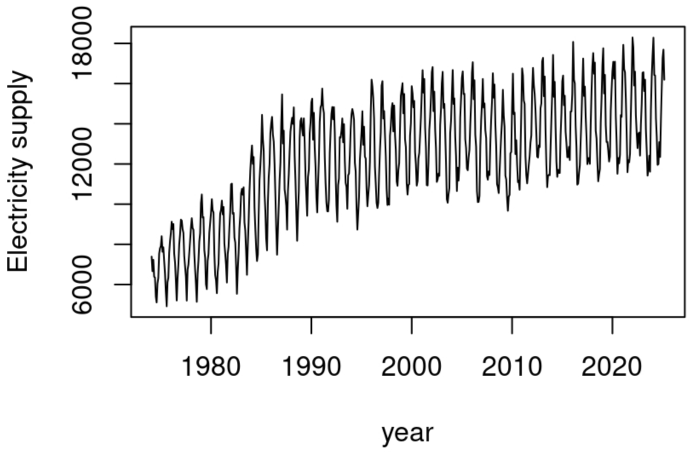

Figure 1: The electricity supply in Sweden.

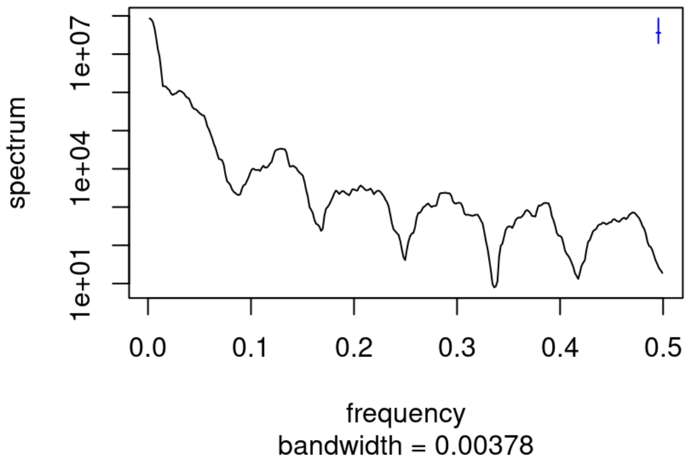

Figure 2: Estimated spectral density, problem 6.

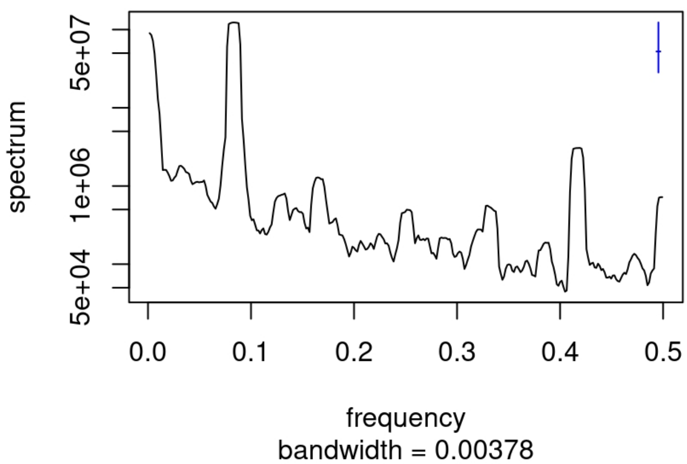

Figure 3: Estimated spectral density, problem 6.

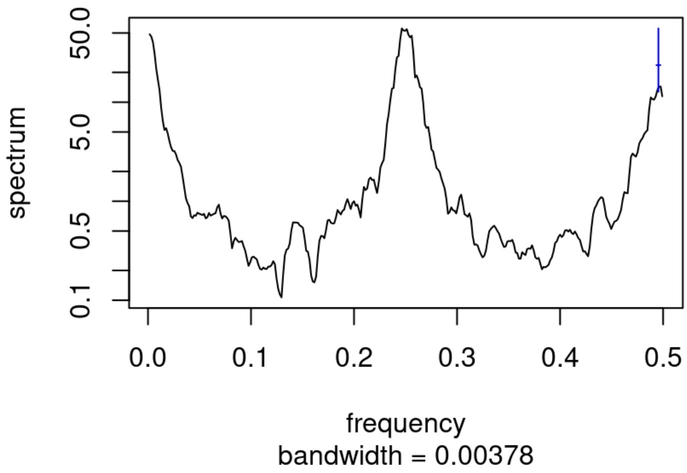

Figure 4: Estimated spectral density, problem 6.

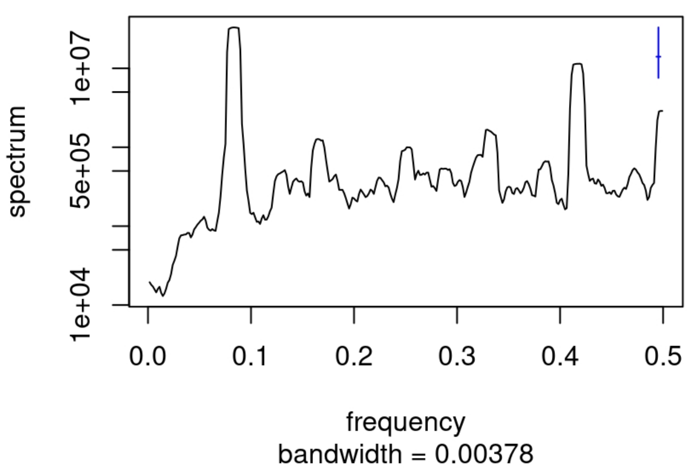

Figure 5: Estimated spectral density, problem 6.