UPPSALA UNIVERSITY
Department of Mathematics
Rolf Larsson

Exam in Mathematical Statistics
Analysis of Time Series 1MS014
2025–08–28
Solutions

1. Let $\{w_t\}$, $t = 0, 1, 2, ...$ be a Gaussian white noise process with $\text{var}(w_t) = 4$ and let
$$x_t = 1.2 + 0.4 w_t w_{t-2} + 0.1 w_{t-2} w_{t-4}.$$
Calculate the mean and autocovariance function of x_t and state whether it is weakly stationary. (5p)

Solution: By independence of the $\{w_t\}$ series and the fact that $E(w_t) = 0$ for all t, the mean function is given by
$$\mu_t = E(x_t) = 1.2 + 0.4E(w_t)E(w_{t-2}) + 0.1E(w_{t-2})E(w_{t-4}) = 1.2.$$
Hence, the autocovariance function may be written as
$$\begin{aligned}
&\gamma(t+h, t) \\
&= \text{cov}(x_{t+h}, x_t) = E\{(x_{t+h} - 1.2)(x_t - 1.2)\} \\
&= E\{(0.4w_{t+h}w_{t+h-2} + 0.1w_{t+h-2}w_{t+h-4})(0.4w_t w_{t-2} + 0.1w_{t-2}w_{t-4})\} \\
&= 0.4^2 E(w_{t+h}w_{t+h-2}w_t w_{t-2}) + 0.4 \cdot 0.1 E(w_{t+h}w_{t+h-2}w_{t-2}w_{t-4}) \\
&+ 0.1 \cdot 0.4 E(w_{t+h-2}w_{t+h-4}w_t w_{t-2}) + 0.1^2 E(w_{t+h-2}w_{t+h-4}w_{t-2}w_{t-4}). \qquad (1)
\end{aligned}$$
Here, by independence and since $E(w_t^2) = \text{var}(w_t) = 4$, the expectations in question are nonzero only if the factors are pairwise equal, so with $I\{A\} = 1$ if A is fulfilled and 0 otherwise, we get
$$\begin{aligned}
E(w_{t+h}w_{t+h-2}w_t w_{t-2}) &= E(w_t^2)E(w_{t-2}^2)I\{h=0\} = 4^2 I\{h=0\}, \\
E(w_{t+h}w_{t+h-2}w_{t-2}w_{t-4}) &= E(w_{t-2}^2)E(w_{t-4}^2)I\{h=-2\} = 4^2 I\{h=-2\}, \\
E(w_{t+h-2}w_{t+h-4}w_t w_{t-2}) &= E(w_t^2)E(w_{t-2}^2)I\{h=2\} = 4^2 I\{h=2\}, \\
E(w_{t+h-2}w_{t+h-4}w_{t-2}w_{t-4}) &= E(w_{t-2}^2)E(w_{t-4}^2)I\{h=0\} = 4^2 I\{h=0\}.
\end{aligned}$$
Hence, (1) yields
$$\begin{aligned}
&\gamma(t+h, t) \\
&= 0.4^2 \cdot 4^2 I\{h=0\} + 0.4 \cdot 0.1 \cdot 4^2 I\{h=-2\} + 0.1 \cdot 0.4 \cdot 4^2 I\{h=2\} \\
&+ 0.1^2 \cdot 4^2 I\{h=0\} \\
&= 2.72 I\{h=0\} + 0.64 I\{|h|=2\} \\
&= \begin{cases} 2.72 & \text{if } h = 0, \\ 0.64 & \text{if } |h| = 2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$
Since the variance $\gamma(t,t) = 2.72$ is finite and none of μ_t or $\gamma(t+h,t)$ is a function of t, we conclude that x_t is weakly stationary.

2. For the ARMA(p, q) models below, where $\{w_t\}$ are white noise processes, find p and q and determine whether they are causal and/or invertible. (6p)

   (a) $x_t = w_t - 0.5w_{t-1}$

   Solution: This is an MA(1) model, i.e. $p = 0$, $q = 1$. It is causal, since all MA models are. It is also invertible, because writing $x_t = \theta(B)w_t$ with $\theta(B) = 1 - 0.5$, $0 = \theta(z) = 1 - 0.5z$ has the solution $z = 2$, which is outside the complex unit circle.

   (b) $x_t = 0.16x_{t-2} + w_t - 0.5w_{t-1}$

   Solution: This is AR(2,1), i.e. $p = 2$, $q = 1$, and we may write $\phi(B)x_t = \theta(B)w_t$ with $\phi(B) = 1 - 0.16B^2$ and $\theta(B) = 1 - 0.5B$ (which have no common roots, se below).

   Invertibility is found as in (a). To check causality, we need to solve $0 = \phi(z) = 1 - 0.16z^2$, i.e. $z^2 = 1/0.16$, which gives us the solutions $z_{1,2} = \pm 1/0.4 = \pm 2.5$. Because $|\pm 2.5| = 2.5 > 1$, these are both outside the complex unit circle, and so the model is causal.

   (c) $x_t = x_{t-1} + w_t - w_{t-1}$

   Solution: Writing the model as $(1 - B)x_t = (1 - B)w_t$, we find that it is equivalent to $x_t = w_t$, i.e. white noise. This means $p = q = 0$, causality and invertibility.

   (d) $x_t = -0.3x_{t-1} + 0.1x_{t-2} + w_t$

   Solution: This is AR(2), i.e. $p = 2$, $q = 0$. It is invertible, since all AR models are. To check causality, we write $\phi(B)x_t = w_t$ with $\phi(B) = 1 + 0.3B - 0.1B^2$ and check the solutions of $0 = \phi(z) = 1 + 0.3z - 0.1z^2$, i.e.

   $$z^2 - 3z - 10 = 0,$$

   which are

   $$z_{1,2} = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 + 10} = \frac{3}{2} \pm \sqrt{\frac{49}{4}} = \frac{3 \pm 7}{2},$$

   i.e. $z_1 = 5$ and $z_2 = -2$. Because both of these are outside the complex unit circle, the model is causal.

3. Let $\{w_t\}$ be a white noise process with variance $\sigma_w^2 = 1$ and define x_t through

$$x_t = w_t + \frac{1}{2}w_{t-1} - \frac{1}{4}w_{t-2} - \frac{1}{8}w_{t-3}.$$

Calculate the autocorrelation function $\rho(h)$ for $h = 1, 2, 3, 4, 5, 6$. (5p)

Solution: At first, we find the autocovariance function $\gamma(h) = \text{cov}(x_{t+h}, x_t)$. Because the w_t are uncorrelated, we get

$$\begin{aligned}
\gamma(0) &= \text{cov}\left(w_t + \frac{1}{2}w_{t-1} - \frac{1}{4}w_{t-2} - \frac{1}{8}w_{t-3}, w_t + \frac{1}{2}w_{t-1} - \frac{1}{4}w_{t-2} - \frac{1}{8}w_{t-3}\right) \\
&= \text{cov}(w_t, w_t) + \left(\frac{1}{2}\right)^2 \text{cov}(w_{t-1}, w_{t-1}) + \left(-\frac{1}{4}\right)^2 \text{cov}(w_{t-2}, w_{t-2}) \\
&+ \left(-\frac{1}{8}\right)^2 \text{cov}(w_{t-3}, w_{t-3}) \\
&= 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 = \frac{85}{64} = 1.328125,
\end{aligned}$$

and similarly,

$$\begin{aligned}
\gamma(1) &= \text{cov}\left(w_{t+1} + \frac{1}{2}w_t - \frac{1}{4}w_{t-1} - \frac{1}{8}w_{t-2}, w_t + \frac{1}{2}w_{t-1} - \frac{1}{4}w_{t-2} - \frac{1}{8}w_{t-3}\right) \\
&= \frac{1}{2}\text{cov}(w_t, w_t) - \frac{1}{4} \cdot \frac{1}{2}\text{cov}(w_{t-1}, w_{t-1}) + \frac{1}{8} \cdot \frac{1}{4}\text{cov}(w_{t-2}, w_{t-2}) \\
&= \frac{1}{2} - \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{4} = \frac{26}{64} = 0.40625,
\end{aligned}$$

$$\begin{aligned}
\gamma(2) &= \text{cov}\left(w_{t+2} + \frac{1}{2}w_{t+1} - \frac{1}{4}w_t - \frac{1}{8}w_{t-1}, w_t + \frac{1}{2}w_{t-1} - \frac{1}{4}w_{t-2} - \frac{1}{8}w_{t-3}\right) \\
&= -\frac{1}{4}\text{cov}(w_t, w_t) - \frac{1}{8} \cdot \frac{1}{2}\text{cov}(w_{t-1}, w_{t-1}) \\
&= -\frac{1}{4} - \frac{1}{8} \cdot \frac{1}{2} = -\frac{20}{64} = -0.3125,
\end{aligned}$$

and

$$\begin{aligned}
\gamma(3) &= \text{cov}\left(w_{t+3} + \frac{1}{2}w_{t+2} - \frac{1}{4}w_{t+1} - \frac{1}{8}w_t, w_t + \frac{1}{2}w_{t-1} - \frac{1}{4}w_{t-2} - \frac{1}{8}w_{t-3}\right) \\
&= -\frac{1}{8}\text{cov}(w_t, w_t) = -\frac{1}{8} = -\frac{8}{64} = -0.125.
\end{aligned}$$

In the same fashion, we find that for $h > 3$,

$$\begin{aligned}
&\gamma(h) \\
&= \text{cov}\left(w_{t+h} + \frac{1}{2}w_{t+h-1} - \frac{1}{4}w_{t+h-2} - \frac{1}{8}w_{t+h-3}, w_t + \frac{1}{2}w_{t-1} - \frac{1}{4}w_{t-2} - \frac{1}{8}w_{t-3}\right) \\
&= 0.
\end{aligned}$$

Conclusively, we get the autocorrelations

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{26}{85} \approx 0.306,$$
$$\rho(2) = \frac{\gamma(2)}{\gamma(0)} = -\frac{20}{85} \approx -0.235,$$
$$\rho(3) = \frac{\gamma(3)}{\gamma(0)} = -\frac{8}{85} \approx -0.094,$$

while for $h > 3$,

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = 0.$$

In particular, $\rho(4) = \rho(5) = \rho(6) = 0$.

4. Consider the process

$$x_t = -0.5x_{t-1} + w_t - 0.9w_{t-1} + 0.2w_{t-2},$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 0.1$. We observe x_t up to time $t = 100$, where the last four observations are $x_{97} = 0.4$, $x_{98} = 0.3$, $x_{99} = 0.2$ and $x_{100} = 0.1$.

(a) Predict the values of x_{101} and x_{102}. Approximations are permitted. (4p)

Solution: We will calculate truncated predictions by using the AR representation $\pi(B)x_t = w_t$. We have

$$(1 - 0.9B + 0.2B^2)w_t = (1 + 0.5B)x_t,$$

which yields

$$\pi(B)(1 - 0.9B + 0.2B^2)w_t = (1 + 0.5B)\pi(B)x_t = (1 + 0.5B)w_t.$$

Hence, with $\pi(z) = 1 + \pi_1 z + \pi_2 z^2 + ...$, we need to solve

$$(1 + \pi_1 z + \pi_2 z^2 + \pi_3 z^3 + ...)(1 - 0.9z + 0.2z^2) = 1 + 0.5z,$$

i.e.

$$1+(\pi_1-0.9)z+(\pi_2-0.9\pi_1+0.2)z^2+(\pi_3-0.9\pi_2+0.2\pi_1)z^2+... = 1+0.5z,$$

which yields

$$\begin{aligned}
\pi_1 &= 0.9 + 0.5 = 1.4, \\
\pi_2 &= 0.9\pi_1 - 0.2 = 1.06, \\
\pi_3 &= 0.9\pi_2 - 0.2\pi_1 = 0.674, \\
\pi_4 &= 0.9\pi_3 - 0.2\pi_2 = 0.3946, \\
\pi_5 &= 0.9\pi_4 - 0.2\pi_3 = 0.22034.
\end{aligned}$$

The truncated predictions become

$$\begin{aligned}
\tilde{x}_{101} &= -\pi_1 x_{100} - \pi_2 x_{99} - ... \\
&\approx -1.4 \cdot 0.1 - 1.06 \cdot 0.2 - 0.674 \cdot 0.3 - 0.3946 \cdot 0.4 = -0.71204
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{x}_{102} \\
&= -\pi_1 \tilde{x}_{101} - \pi_2 x_{100} - \pi_3 x_{99} - ... \\
&\approx 1.4 \cdot 0.71204 - 1.06 \cdot 0.1 - 0.674 \cdot 0.2 - 0.3946 \cdot 0.3 - 0.22034 \cdot 0.4 \\
&= 0.54954.
\end{aligned}$$

(b) Calculate 95% prediction intervals for x_{101} and x_{102}. (3p)

Solution: The mean square predicion error m steps ahead is given by $\sigma_w^2 \sum_{j=1}^{m-1} \psi_j^2$, where the ψ_j are the coefficients in the MA representation, with $\psi_0 = 1$. We only need to find ψ_1. To this end, $x_t = \psi(B)w_t$ implies

$$\begin{aligned}\psi(B)(1+0.5B)x_t &= (1-0.9B+0.2B^2)\psi(B)w_t \\ &= (1-0.9B+0.2B^2)x_t,\end{aligned}$$

so that with $\psi(z) = 1 + \psi_1 z + ...$, we have

$$(1+\psi_1 z + ...)(1+0.5z) = 1 - 0.9z + 0.2z^2,$$

implying $\psi_1 + 0.5 = -0.9$, i.e. $\psi_1 = -1.4$.
With $\sigma_w^2 = 0.1$, this gives the 95% prediction interval for x_{101} as

$$-0.71204 \pm 1.96\sqrt{0.1} = -0.71204 \pm 0.61981 = (-1.332,\ -0.092),$$

and for x_{102}, we find the corresponding interval

$$\begin{aligned}0.54954 \pm 1.96\sqrt{0.1(1+1.4^2)} &= 0.54954 \pm 1.06636 \\ &= (-0.517,\ 1.616).\end{aligned}$$

5. A time series $\{x_t\}$ follows the model

$$x_t = 0.5x_{t-1} + w_t,$$

where $\{w_t\}$ is normally distributed white noise with variance $\sigma_w^2 = 2$. This series is used as input for constructing

$$y_t = 0.6y_{t-1} + x_t,$$

for $t = 0, \pm 1, \pm 2, ...$

(a) Calculate the spectral densitiy of x_t at the frequencies $\omega = 0.1$ and $\omega = 0.4$. (2p)

Solution: We have $\phi(B)x_t = w_t$ with $\phi(B) = 1 - 0.5B$. We want to use the formula

$$f_x(\omega) = \frac{\sigma_w^2}{|\phi(e^{-2\pi i\omega})|^2}.$$

Here, $\sigma_w^2 = 2$ and

$$\begin{aligned}|\phi(e^{-2\pi i\omega})|^2 &= |1 - 0.5e^{-2\pi i\omega}|^2 = (1 - 0.5e^{-2\pi i\omega})(1 - 0.5e^{2\pi i\omega}) \\ &= 1 + 0.5^2 - \frac{e^{2\pi i\omega} + e^{-2\pi i\omega}}{2} = 1.25 - \cos(2\pi\omega).\end{aligned}$$

Hence,

$$f_x(\omega) = \frac{2}{1.25 - \cos(2\pi\omega)},$$

and insertion gives $f_x(0.1) \approx 4.535$ and $f_x(0.4) \approx 0.971$.

(b) Calculate the spectral densitiy of y_t at the frequencies $\omega = 0.1$ and $\omega = 0.4$. (3p)

Solution: We will find the filter $y_t = \sum_j a_j x_{t-j}$ and then use the frequency response function. To find the filter, we use recursion to get

$$\begin{aligned} y_t &= 0.6y_{t-1} + x_t = 0.6(0.6y_{t-2} + x_{t-1}) + x_t = 0.6^2 y_{t-2} + 0.6x_{t-1} + x_t \\ &= ... = \sum_{j=0}^{\infty} 0.6^j x_{t-j}, \end{aligned}$$

i.e. $a_j = 0.6^j$ for $j = 0, 1, 2, ...$ and 0 for $j < 0$. Hence, we have the frequency response function

$$A(\omega) = \sum_{j=0}^{\infty} a_j e^{-2\pi i\omega j} = \sum_{j=0}^{\infty} (0.6e^{-2\pi i\omega})^j = \frac{1}{1 - 0.6e^{-2\pi i\omega}}.$$

This means that

$$\begin{aligned} |A(\omega)|^2 = A(\omega)\overline{A(\omega)} &= \frac{1}{(1 - 0.6e^{-2\pi i\omega})(1 - 0.6e^{2\pi i\omega})} \\ &= \frac{1}{1.36 - 1.2\cos(2\pi\omega)}. \end{aligned}$$

Hence, the formula

$$f_y(\omega) = |A(\omega)|^2 f_x(\omega)$$

yields $f_y(0.1) \approx 11.65$ and $f_y(0.4) \approx 0.417$.

(c) Compare and discuss your results. (1p)

Solution: The filter in (b) is recursive, with relatively high weights, meaning that is 'smoothes out' the series relatively much. Hence, it is a low-pass filter, decreasing the weights for the high frequencies relative to the low ones. This is what we see from our numbers, where the spectral density at 0.4 decreases after filtering, while it increases at 0.1.

6. Four time series with $n = 100$ observations each were simulated according to the models

$$
\begin{aligned}
x_t &= 0.9x_{t-1} + w_t, \quad (1)\\
x_t &= -0.9x_{t-1} + w_t, (2)\\
x_t &= 0.9x_{t-4} + w_t, \quad (3)\\
x_t &= 0.9x_{t-12} + w_t, \quad (4)
\end{aligned}
$$

where $\{w_t\}$ was normally distributed white noise with variance $\sigma_w^2 = 1$.

For three of these series, the corresponding spectral densities were estimated (nonparametric, `spans=4`) , and they are depicted in figures 1-3.

Match three of the models (1)-(4) with the figures 1-3. Motivate your answer.

(5p)

Solution: The graph of figure 1 appears to be fairly non periodic with highest weights at high frequencies. This would correspond to a series that oscillates up and down relatively rapidly, and such a one is generated by model (2).

Figure 2 depicts a periodic graph, with heigh weights on frequencies that are multiples of about 0.08, which corresponds to a period of $1/0.08 \approx 12$. This corresponds to model (4) (a seasonal AR with period 12).

Finally, the graph in figure 3 somewhat mirrors the one in figure 1, and gives highest weights on low frequencies without showing too much of periodicity. This should correspond to model (1), that generates a fairly smooth series.

7. Consider the system

$$x_{1,t} = x_{1,t-1} + w_{1,t},$$
$$x_{2,t} = 0.8x_{1,t-1} + w_{2,t},$$

where $\{w_{1,t}\}$ and $\{w_{2,t}\}$ are independent white noise processes.

(a) Argue that $x_{1,t}$ and $x_{2,t}$ are not stationary. (3p)

Solution: The first equation implies that $x_{1,t}$ is a random walk, hence its variance increases linearly with t. This means that $x_{1,t}$ is not stationary. By the second equation, $x_{2,t}$ is a linear function of $x_{1,t-1}$. Hence, the variance of $x_{2,t}$ must also increase linearly in t. Conclusively, $x_{2,t}$ cannot be stationary.

(b) Find a constant a such that $x_{1,t} - ax_{2,t}$ is stationary. (3p)

Solution: It follows that

$$x_{1,t} - ax_{2,t} = (1 - 0.8a)x_{1,t-1} + w_{1,t} - aw_{2,t}.$$

Hence, with $1 - 0.8a = 0$, i.e. $a = 1/0.8 = 5/4 = 1.25$, the first term cancels and we get a white noise (hence stationary) process $w_{1,t} - 1.25w_{2,t}$. Thus, the answer is $a = 5/4 = 1.25$.

Appendix: figures

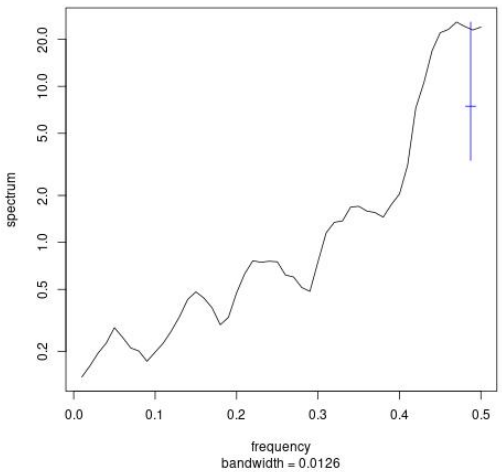

Figure 1: Estimated spectral density, problem 6.

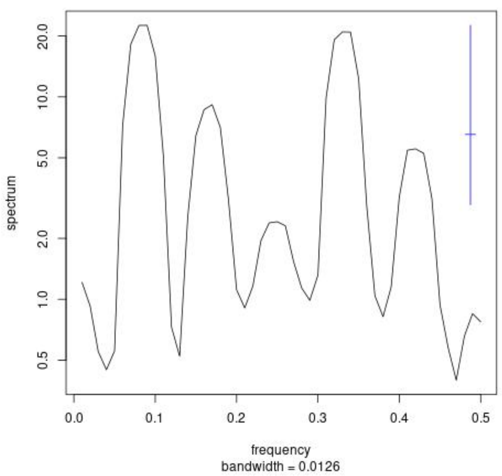

Figure 2: Estimated spectral density, problem 6.

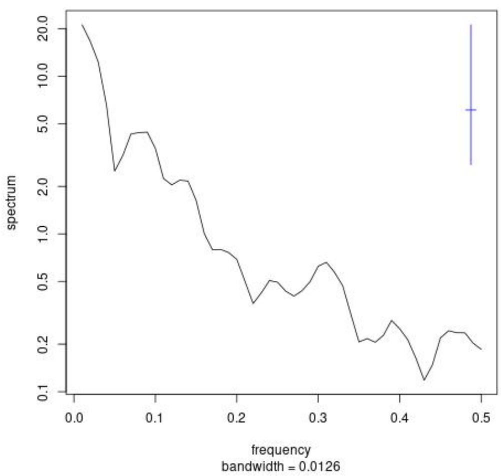

Figure 3: Estimated spectral density, problem 6.