Uppsala Universitet
Matematiska Institutionen
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Prov i matematik
Elementär talteori
2017-03-08

Skrivtid: 8.00 – 13.00.
Tillåtna hjälpmedel: Papper, skrivdon och miniräknare.

1. Solve the Diophantine equations
(a) $24x + 15y - 25z = 2$.
(b) $21x + 14y - 56z = 2$.
(5p)

2. Determine the zeros of the following polynomials:
(a) $X^3 + X^2 + 3$ in $\mathbb{Z}_{125}$;
(b) $X^2 - 3X$ in $\mathbb{Z}_{221}$;
(5p)

3. Determine whether the following residue classes are squares:
(a) $\overline{435}$ in $\mathbb{Z}_{607}$.
(b) $\overline{616}$ in $\mathbb{Z}_{435}$.
(5p)

4. (a) Prove that $\overline{2}$ is a primitive root in $\mathbb{Z}_{29}^{\times}$.
(b) Determine the zeros of the polynomial $X^{64} - \overline{16}$ in $\mathbb{Z}_{29}$.
(5p)

5. Show that the only integer solution to the equation

$$5x^3 + 7y^3 = 11z^3$$

is $x = y = z = 0$. (Hint: First consider the equation mod m, for a suitable choice of m.)
(5p)

6. (a) Find the continued fraction expansion of $\sqrt{7}$ and compute its first four convergents.
(b) Find three solutions $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ to the equation $x^2 - 7y^2 = 1$.
(c) Are there any solutions $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ to $x^2 - 7y^2 = -1$?
(5p)

7. For any positive integer n, let $\Omega(n) = \sum_{p|n} \mathrm{ord}_p(n)$ (this is the total number of primes appearing in the prime factorization of n, counting multiplicity) and $\lambda(n) = (-1)^{\Omega(n)}$. Prove that $\lambda(n)$ is totally multiplicative, and that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a perfect square} \\ 0 & \text{otherwise.} \end{cases}$$

(5p)

8. Find all positive integers n such that $\phi(n) \mid n$.

(5p)

LYCKA TILL / GOOD LUCK!