Tentalösningar

1) $f(x,y) = xy^2 + yx^2 - x$

Stationära punkter ges av $\nabla f = (0,0)$

$$\nabla f(x,y) = (\underbrace{y^2 + 2xy - 1}_{①}, \underbrace{x^2 + 2xy}_{②}) = (0,0)$$

$$x^2 + 2xy = x(x+2y) \underset{②}{=} 0 \iff x=0 \text{ eller } x=-2y$$

$$\underline{x=0} \underset{①}{\Rightarrow} y^2 = 1 \iff (0,1), (0,-1)$$

$$\underline{x=-2y} \underset{①}{\Rightarrow} y^2 + 2(-2y)y - 1 = 0 \iff -3y^2 - 1 = 0$$

som saknar lösning

De stationära punkterna är $(0,1), (0,-1)$

Vi beräknar Hessianen.

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2y & 2y+2x \\ 2y+2x & 2x \end{bmatrix}$$

$$H(0,1) = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} \qquad \det H(0,1) = -4 \quad - \text{ sadelpunkt}$$

$$H(0,-1) = \begin{bmatrix} -2 & -2 \\ -2 & 0 \end{bmatrix} \qquad \det H(0,-1) = -4 \quad \text{sadelpunkt}$$

2) Finn största och minsta värde för

$$f(x,y) = x^2 - y^3 \quad \text{på} \quad x^2 + y^2 \leq 1 .$$

Extrema finns eftersom f är kont. och området kompakt

Stationära punkter i det inre $\quad \nabla f = (2x, -3y^2) = (0,0)$

$$\iff (x,y) = (0,0)$$

– bara origo.

$$\boxed{f(0,0) = 0}$$

På randen: Lagrange metod säger att

$$\nabla f \parallel \nabla g \quad (\text{där } g(x,y) = x^2 + y^2)$$

dvs

$$\det \begin{bmatrix} 2x & -3y^2 \\ 2x & 2y \end{bmatrix} = 0$$

$$\iff 4xy + 6xy^2 = 0 \iff 2xy + 3xy^2 = 0$$

$$\iff xy(2 + 3y) = 0 \quad \text{så} \quad x = 0 \text{ eller } y = 0$$

eller $(2 + 3y) = 0$.

$$x = 0 \underset{x^2+y^2=1}{\implies} y^2 = 1 \implies y = \pm 1 \quad (0,1) \quad (0,-1)$$

$$y = 0 \implies x^2 = 1 \implies x = \pm 1 \quad (1,0) \quad (-1,0)$$

$$2 + 3y = 0 \implies y = -\frac{2}{3} \underset{x^2+y^2=1}{\implies} x^2 = 1 - \frac{4}{9} \implies x = \pm \frac{\sqrt{5}}{3}$$

$$\left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\right) \quad \left(-\frac{\sqrt{5}}{3}, -\frac{2}{3}\right)$$

$f(0,1) = -1$ $\quad$ $f(1,0) = 1$

$f(0,-1) = +1$ $\quad$ $f(-1,0) = 1$

$$f\left(\pm\frac{\sqrt{5}}{3}, -\frac{2}{3}\right) = \frac{5}{9} + \frac{8}{27} = \frac{15}{27} + \frac{8}{27} = \frac{23}{27}$$

f:s största värde är 1 och antas i (0,-1)
(1, 0)
och (-1, 0)

& minsta värdet är -1 och antas i (0, 1)

Alternativa lösningar: Randen kan parametriseras direkt, antingen med y (bitvis) eller polärt, med "θ".

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3) $u = \frac{1}{\sqrt{2}}(x-y) \qquad v = \frac{1}{\sqrt{2}}(x+y)$

$$\begin{cases} \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial \phi}{\partial u}\left(\frac{1}{\sqrt{2}}\right) + \frac{\partial \phi}{\partial v}\cdot\left(\frac{1}{\sqrt{2}}\right) \\ \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial \phi}{\partial u}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\partial \phi}{\partial v}\left(\frac{1}{\sqrt{2}}\right) \end{cases}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial v}\right) = \frac{1}{\sqrt{2}}\left(\frac{\partial^2 \phi}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 \phi}{\partial v \partial u}\frac{\partial v}{\partial x} + \frac{\partial^2 \phi}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 \phi}{\partial v^2}\frac{\partial v}{\partial x}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\partial^2 \phi}{\partial u^2}\frac{1}{\sqrt{2}} + \frac{\partial^2 \phi}{\partial u \partial v}\frac{1}{\sqrt{2}} + \frac{\partial^2 \phi}{\partial v \partial u}\frac{1}{\sqrt{2}} + \frac{\partial^2 \phi}{\partial v^2}\cdot\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(\frac{\partial^2 \phi}{\partial u^2} + 2\frac{\partial^2 \phi}{\partial u \partial v} + \frac{\partial^2 \phi}{\partial v^2}\right)$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial v}\right) = \frac{1}{\sqrt{2}}\left(-\frac{\partial^2 \phi}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 \phi}{\partial v \partial u}\frac{\partial v}{\partial y} + \frac{\partial^2 \phi}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 \phi}{\partial v^2}\frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial^2 \phi}{\partial u^2} - 2\frac{\partial^2 \phi}{\partial u \partial v} + \frac{\partial^2 \phi}{\partial v^2}\right)$$

$$\Longrightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}$$

$\square$

4) E : $xyz + x^2y^3 = 2$

a) Linjärisera VL : Låt $f(x,y,z) = xyz + x^2y^3$

$f(1,1,1) = 2$ $\nabla f(x,y,z) = (yz + 2xy^3, xz + 3x^2y^2, xy)$

så $\nabla f(1,1,1) = (3,4,1)$

Vi får en linjäriserad ekvation som är

$2 + (3,4,1)(x-1, y-1, z-1) = 2$

$\iff \cancel{3,4}\ 3(x-1) + 4(y-1) + (z-1) = 0$

Svar: Tangentplanets ekvation är $3x + 4y + z = 8$

b) Eftersom $\frac{\partial f}{\partial x} = 3 \neq 0$ $\frac{\partial f}{\partial y} = 4 \neq 0$ & $\frac{\partial f}{\partial z} = 1 \neq 0$

kan alla variablerna uttryckas som funktioner av de andra nära $(1,1,1)$, enligt implicita funktionsatsen.

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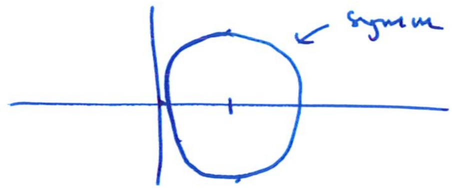
$$\iint_D (x^2+y^2-2y)\,dx\,dy = \iint_D (x^2+y^2)\,dx\,dy$$

y udda i y
D symm.

$$D = \{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+y^2\le 1\}$$



symm i y!

$$= \left\{\begin{array}{ll} x = 1 + r\cos\theta & 0\le r\le 1 \\ y = r\sin\theta & 0\le\theta\le 2\pi \end{array}\right\}$$

(Jacobianen blir samma som för vanliga polära)

$$= \int_0^{2\pi}\int_0^1 \left((1+r\cos\theta)^2 + (r\sin\theta)^2\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(1 + r^2\cos^2\theta + 2r\cos\theta + r^2\sin^2\theta\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(1 + r^2 + \underbrace{2r\cos\theta}_{\text{integrera } \cos\theta \text{ över en hel period ger } 0}\right) r\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^1 r + r^3\,dr\,d\theta = 2\pi\left[\frac{r^2}{2} + \frac{r^4}{4}\right]_0^1 =$$

$$= 2\pi\left(\frac{1}{2}+\frac{1}{4}\right) = \frac{6\pi}{4} = \underline{\underline{\frac{3\pi}{2}}}$$

⑥ $\bar{F}(x,y,z) = (xz^2, x^2y, y^2z-1)$

$\nabla\cdot\bar{F} = z^2+x^2+y^2$

(a) Flodet ut ur D ar enligt divergensatsen

$$VL = \iint_{\partial D} \bar{F}\cdot d\bar{S} = \iiint_{D} \nabla\cdot\bar{F}\, dV = HL$$

sfäriska koordinater

och $HL = \iiint_{D} x^2+y^2+z^2 \, dxdydz = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} r^4 \sin\varphi \, dr\, d\varphi\, d\theta$

$$= 2\pi \cdot \int_0^{2}\int_0^{\pi/2} r^4 \sin\varphi \, dr\, d\varphi = 2\pi\left[\frac{r^5}{5}\right]_0^2 \cdot \left[-\cos\varphi\right]_0^{\pi/2}$$

$$= 2\pi\cdot\frac{32}{5}\cdot 1 = \underline{\underline{\frac{64}{5}\pi}}$$

(b) $$VL = \iint_{\partial D} \bar{F}\cdot d\bar{S} = \iint_{\text{halvsfären}} \bar{F}\cdot d\bar{S} + \iint_{\text{bottenplattan}} \bar{F}\cdot d\bar{S}.$$

$$\iint_{\text{bottenplattan}} \bar{F}\cdot d\bar{S} = \iint_{\text{bottenplattan}} (xz^2, x^2y, y^2z-1)\cdot(0,0,-1)\, dxdy$$

bottenplattan: $\left\{\begin{matrix} x^2+y^2\le 4 \\ z=0 \end{matrix}\right\}$

$$= \iint_{\text{bottenplattan}} (0, x^2y, -1)\cdot(0,0,-1)\, dxdy = \iint dxdy = \text{Area}\left\{x^2+y^2\le 4\right\}$$

$$= \pi\cdot 2^2 = 4\pi$$

$$\Rightarrow \iint_{\text{halv-sfären}} \vec{F} \cdot d\vec{S} = \frac{64\pi}{5} - 4\pi = \frac{64\pi}{5} - \frac{20\pi}{5} = \frac{44\pi}{5}$$

~~Så~~ $\frac{44\pi}{5} > 4\pi$ så flödet är störst genom halvsfären.

⑦ $\bar{F}(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z^2 \right)$

C – skärningskurvan mellan $z = 1+y^2$

och $x^2+y^2 = 4$

(a) $\nabla \times \bar{F} = \bar{0}$ ty

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & z^2 \end{vmatrix} = \left(0, 0, \frac{1\cdot(x^2+y^2)-2x^2}{(x^2+y^2)^2} + \frac{1\cdot(x^2+y^2)-2y^2}{(x^2+y^2)^2} \right)$$

$$= (0, 0, 0)$$

(b) $\int_C \bar{F} \cdot d\bar{r} = 2\pi$?

$(0, 0, z^2)$ är konservativt (med potential $\phi(x,y,z) = \frac{z^3}{3}$)

så det ger inget bidrag till kurvintegralen.

<u>Alltså</u>: $\int_C \bar{F} \cdot d\bar{r} = \int_C \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right) \cdot \left(-2\sin\theta, 2\cos\theta, --- \right) d\theta$

$$= \int_0^{2\pi} \left(-\frac{2\sin\theta}{4}, \frac{2\cos\theta}{4}, 0 \right) \cdot \left(-2\sin\theta, 2\cos\theta, - \right) d\theta$$

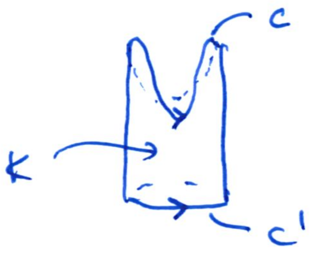
parametrisera kurvan med $\theta \overset{\bar{r}}{\mapsto} (2\cos\theta, 2\sin\theta, (1+4\sin^2\theta)^2)$

$\frac{d\bar{r}}{d\theta} = (-2\sin\theta, 2\cos\theta, ---)$

$$= \int_0^{2\pi} d\theta = \underline{\underline{2\pi}}$$

Alt 2 : Kurvan C och cirkeln $\overbrace{x^2+y^2=4}^{C'}$ i xy-planet

begränsar en del K av cylindern:



Stokes sats ger då $\displaystyle\int_C \bar{F}\cdot d\bar{r} - \int_{C'} \bar{F}\cdot d\bar{r} = \int_K \nabla\times\bar{F}\cdot d\bar{S} = 0$

så $\displaystyle\int_C \bar{F}\cdot d\bar{r} = \int_{C'} \bar{F}\cdot d\bar{r} = \int_{C'} \Big(-\underbrace{\frac{y}{x^2+y^2}}_{\frac{2\sin\theta}{4}}, \underbrace{\frac{x}{x^2+y^2}}_{\frac{2\cos\theta}{4}}, 0\Big)\cdot(-2\sin\theta, 2\cos\theta, 0)\, d\theta$

på C' är $z=0$

$$= \int_0^{2\pi} d\theta = 2\pi.$$

Alt 3: Direkt parametrisering av C funkar också.

$\theta \xrightarrow{\bar{r}} (2\cos\theta, 2\sin\theta, (1+4\sin^2\theta))$ (som i Alt 1)

$\displaystyle\frac{d\bar{r}}{d\theta} = (-2\sin\theta, 2\cos\theta, 8\sin\theta\cos\theta)$

och $\bar{F} = \left(-\frac{1}{2}\sin\theta, \frac{1}{2}\cos\theta, 1+8\sin^2\theta+16\sin^4\theta\right)$

så $\displaystyle\int_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi} \underbrace{\sin^2\theta+\cos^2\theta}_{1} + 8\sin\theta\cos\theta + 64\sin^3\theta\cos\theta + 128\sin^5\theta\cos\theta \, d\theta$

$$= 2\pi + \Big[4\sin^2\theta\Big]_0^{2\pi} + \Big[16\sin^4\theta\Big]_0^{2\pi} + \Big[\frac{128}{6}\sin^6\theta\Big]_0^{2\pi}$$

$$= \underline{2\pi}$$

8) $\begin{cases} x' = 2x + y \\ y' = x + 2y \end{cases}$

Koefficientmatrisen är $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Egenvärdena bestäms: $\det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0$

$\iff (2-\lambda)^2 - 1 = 0 \iff \lambda^2 - 4\lambda + 4 - 1 = \lambda^2 - 4\lambda + 3 = 0$

$\iff \lambda = 1$ eller $\lambda = 3$.

Egenvektorerna, $\lambda = 1$; $\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

så en egenvektor är $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ tex $\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$

$\lambda = +3$:

$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ så en egenvektor är $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Allmänna lösningen är

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} + B \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \qquad A, B \in \mathbb{R}.$$