1MA183 Multivariate calculus M
Uppsala University

Exam
1.6.2022
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Writing time 8-13. Equipment: only writing equipment (formulas are provided on page 3). You may answer in Swedish or in English. Solutions must be provided with justifications. Each correct answer gives at most 5 points. For grades 3,4, 5 you need 16, 25, and 32 points, respectively. På svenska, se sidan 2.

1. Is the function $f(x,y) = \frac{x^2y^2}{|x|y^2}$ bounded on $\{(x,y) : x^2+y^2 \leq 1, (x,y) \neq (0,0)\}$? (5p)

   Solution sketch. The idea is to check what happens on the bad points when f is not defined. These are $x=0$ or $y=0$. Hence one should check that in these bad points values does not explode, i.e. when we approach bad points. Now we observe that function reduces to $f(x,y) = |x|$ from which we see clearly that it is bounded.

2. Find the largest value of $f(x,y) = \frac{x^2+y^2}{(2+x^2+y^2)^2}$ in $x \in \mathbb{R}$, $\quad -1 \leq y \leq 1$. (5p)

   Solution sketch. To obtain largest value one needs to check extreme points where the gradient is zero and what happens at the boundary $|y| = 1$ and at infinity when $|x| \to \infty$. There was many ways to do this and tiny numerical errors did not matter. For full points however all this has to be acknowledged somehow.

3. Determine all functions f of one variable such that $u(x,y) = f(x^2+y^2)$ solves the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for $(x,y) \neq (0,0)$. (5p)

   Solution sketch. This was a direct placement: just compute partials of u which then leads to the equation for f:

$$2f'(x^2+y^2) + 4x^2f''(x^2+y^2) + 2f'(x^2+y^2) + 4y^2f''(x^2+y^2) = 0.$$

   This further simplifies into, with $u = x^2+y^2$,

$$f'(u) + uf''(u) = 0.$$

   Noting that $(uf'(u))' = f'(u) + uf''(u)$ we can integrate once to get

$$uf'(u) = C$$

   that leads finally to

$$f(u) = C \log u + D.$$

4. Let $f(x,y,z) = \sin(xy) + z^2$. (5p)

   (a) Find the direction of most decrease at point $(x_0, y_0, z_0) = (0,0,1)$.

   (b) Compute the directional derivative into the direction $v = (1,1,1)$ at the point $(x_0,y_0,z_0) = (0,0,1)$.

   Solution sketch. This was a straightforward computation. In the a-part one needs to compute the gradient ∇f at $(0,0,1)$ and observe that direction of most decrease is $-\nabla f$. For the b-part one computes inner product $\nabla f \cdot v$, where v is now normalised! Normalisation was required for full points.

5. Show that the equation $y^{10} - y = x - 1$ has a solution $y = y(x)$ that is close to 0 whenever x is close to 1. Find the coefficients a_0, a_1, and a_2 in the Taylor approximation

$$y(x) \approx a_0 + a_1 x + a_2 x^2$$

in the neighbourhood of $x = 1$. (5p)

Solution sketch. This is a direct application of implicit function theorem. For $F(x, y) = 0$ if $F'_y \neq 0$ at $(1, 0)$, then one can solve $y = y(x)$ around that point. Then the derivatives can be computed from implicit differentiation. First differentiating $y^{10} - y - x - 1 = 0$ in x (and assuming $y = y(x)$) to get

$$y^9 y' - y' - 1 = 0.$$

From this plugging $y = 0$ one gets $y'(0)$. Differentiating again one can then solve for $y''(0)$ similarly, from which one can get coefficients a_0, a_1, a_2 from Taylor expansion

$$y - y(0) = y'(0)(x - 1) + \frac{y''(0)}{2}(x - 1)^2.$$

6. Compute $\int\int_D x - y dx dy$, where $D = \{(x, y) : x^2 + \frac{y^2}{4} \leq 1\}$. (5p)

Solution sketch. Direct integration that can be solved either directly or by using transformation $x = r\cos\varphi$, $y = 2r\sin\varphi$. Note that basic polar coordinates does not do the trick!

7. Compute the curve integral $\int_\gamma \left(x^{y-1}y + y\right) dx + \left(x^y \log x + x\right) dy$, where γ is the curve $\mathbf{r}(t) = (x(t), y(t)) = (2 + \sin t, 2 - \cos t)$, $\quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. (5p)

Solution sketch. This is an exercise for applying Green's formula. The problem can be solved either by observing that we have potential field and then a) compute the potential function or, b) use the fact that we have path independence and then choose a nicer path. Another equivalent option is to close the curve with a simple path (for which path integral can be solved) and then applying the Green's formula.

8. Compute $\int_{\mathbb{R}} e^{-x^2} dx$. (5p)

Solution sketch. The idea is to use 2-dimensional integral

$$\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 = \int_{\mathbb{R}^2} e^{-x^2 - y^2} dx dy$$

that is now easy to compute in polar coordinates and one gets π. Then taking the square root yields the result. Trying to compute one-dimensional integrals directly did not give points.

Skrivtid 8-13. Utrustning: endast skrivutrustning (formler finns på sidan 3). Du kan svara på svenska eller engelska. Lösningarna skall vara försedda med motiveringar. Varje korrekt löst uppgift ger högst 5 poäng. För betygen 3,4, 5 krävs minst 16, 25 respektive 32 poäng.

1. Är funktionen $f(x,y) = \frac{x^2y^2}{|x|y^2}$ begränsad i området $\{(x,y) : x^2+y^2 \leq 1, (x,y) \neq (0,0)\}$? (5p)

2. Bestäm den största värdet av $f(x,y) = \frac{x^2+y^2}{(2+x^2+y^2)^2}$ i området $x \in \mathbb{R}, \quad -1 \leq y \leq 1$. (5p)

3. Bestäm alla funktioner f av en variabel för vilka $u(x,y) = f(x^2+y^2)$ löser Laplace-ekvationen $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ för $(x,y) \neq (0,0)$. (5p)

4. Låt $f(x,y,z) = \sin(xy) + z^2$. (5p)
   (a) Hitta riktningen för den största minskningen i punkten $(x_0,y_0,z_0) = (0,0,1)$.
   (b) Beräkna riktningsderivatan i riktning $v = (1,1,1)$ i punkten $(x_0,y_0,z_0) = (0,0,1)$.

5. Visa att ekvationen $y^{10} - y = x - 1$ har en lösning $y = y(x)$ som är nära 0 när x är nära 1. Hitta koefficienterna a_0, a_1 och a_2 i Taylor-utvecklingen
$$y(x) \approx a_0 + a_1 x + a_2 x^2$$
i en omgiving av $x = 1$. (5p)

6. Beräkna $\int\int_D x - y dxdy$, där $D = \{(x,y) : x^2 + \frac{y^2}{4} \leq 1\}$. (5p)

7. Beräkna kurvintegralen $\int_\gamma \left(x^{y-1}y + y\right) dx + (x^y \log x + x)\, dy$, där γ är kurvan $\mathbf{r}(t) = (x(t), y(t)) = (2+\sin t, 2-\cos t), \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. (5p)

8. Beräkna $\int_{\mathbb{R}} e^{-x^2} dx$. (5p)

Formulas:

- $(x, y) = (r\cos\varphi, r\sin\varphi)$, $(x, y, z) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$
- $f(x+h) = f(x) + Ah + h\rho(h)$
- $f'_v = \nabla f \cdot v$
- Jacobian $(J)_{ij} = \frac{\partial f_i}{\partial x_j}$. $\int\int_D f(x,y)dxdy = \int\int_E f(u,v)|J|dudv$.
- $\int\int f(x,y)dxdy = \int\left[\int f(x,y)dx\right]dy = \int\left[\int f(x,y)dy\right]dx$
- $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}dt$
- $\int_\gamma f\mathbf{t}ds = \int_a^b f(\mathbf{r}(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}dt$
- $A = \int\int_D |\mathbf{r}'_s \times \mathbf{r}'_t|dsdt$
- $\int\int_\Gamma fdS = \int\int_D f(\mathbf{r}(s,t))|\mathbf{r}'_s \times \mathbf{r}'_t|dsdt$
- potential field (potentialfält) $(P, Q, R) = \nabla U$
- $\text{div}(P, Q, R) = \partial_x P + \partial_y Q + \partial_z R$
- $\text{rot}(P, Q, R) = (\partial_y R - \partial_z Q, \partial_z P - \partial_x R, \partial_x Q - \partial_y P)$
- Green: $\int_{\partial D} \mathbf{F}\cdot\mathbf{t}ds = \int\int_D \text{rot}\mathbf{F}\cdot\mathbf{n}dxdy$
- Stokes: $\int_{\partial\Gamma} \mathbf{F}\cdot\mathbf{t}ds = \int\int_\Gamma \text{rot}\mathbf{F}\cdot\mathbf{n}dS$
- Gauss: $\int\int_{\partial K} \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_K \text{div}\mathbf{F}dxdydz$
- $\sin(2\varphi) = 2\cos\varphi\sin\varphi$, $\cos(2\varphi) = \cos^2\varphi - \sin^2\varphi$