Exam - Fourier analysis

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Exam in Fourier Analysis, 5 credits
1MA211
Kandfy, Kandma, Fristående

Writing time: 14:00–19:00. Allowed aids: writing materials, table of formulæ.

There are 8 problems in this exam. For the grades 3, 4 and 5 you should obtain at least 18, 25 and 32 points respectively. You have to motivate every step in your to get the full score from a question.

1. Let f be a 2π periodic function with $f(t) = \pi^2 - t^2$ for $-\pi < t \leq \pi$.
   (a) Find the Fourier series of trigonometric form.
   (b) To which function does the Fourier series converge? Motivate your answer!
   (c) Using the result in a) calculate the series
   $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$
   (d) Calculate the series
   $$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

   8 points

2. Consider the ordinary differential equation
   $$-u''(x) + u(x) = f(x)$$
   where $f \in L^1(\mathbb{R})$. Show that the solution can be written as
   $$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{1+\xi^2} e^{ix\xi} \, d\xi$$

   4 points

3. Let $f(x) = e^{-x^2}$. Calculate the convolution $g(x) := f * f(x)$. Note that the solution has to be written on explicit form for full points. Integral representation is not enough.

   4 points

4. Calculate c_0, c_1, so that the integral
   $$\int_0^{\infty} |c_0 + c_1 x - \sin x|^2 e^{-x} \, dx$$
   is minimised. Hint: use Gram-Schmidt orthogonalisation.

   4 points

5. Solve the following boundary value problem for the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & 0 < x < \pi, t > 0 \\ u(0,t) = -\pi, u(\pi,t) = 2\pi, & t > 0 \\ u(x,0) = 0, & 0 < x < \pi \end{cases}$$

5 points

6. Use Laplace transform to solve the differential equation

$$\begin{cases} y'' + 5y' - 24y = 28e^{-x} \\ y(0) = 6, y'(0) = 0 \end{cases}$$

5 points

7. Define the integral operator

$$Tf(x) = \int_{-\infty}^{\infty} \frac{2\sin(x-y)}{x-y} f(y)\,\mathrm{d}y.$$

Show that there exists a constant $C > 0$ such that

$$\|Tf\|_{L^2(\mathbb{R})} \leq C \|f\|_{L^2(\mathbb{R})}.$$

Hint: Use Plancherel's formulæ.

5 points

8. Let $f \in \mathcal{C}^2(\mathbb{T})$.

(a) Prove that for some constant $C > 0$

$$\left|\widehat{f}(n)\right| \leq \frac{C}{n^2}.$$

(b) Is the Fourier series of $f \in \mathcal{C}^2(\mathbb{T})$ uniformly convergent? Prove your claim!

5 points