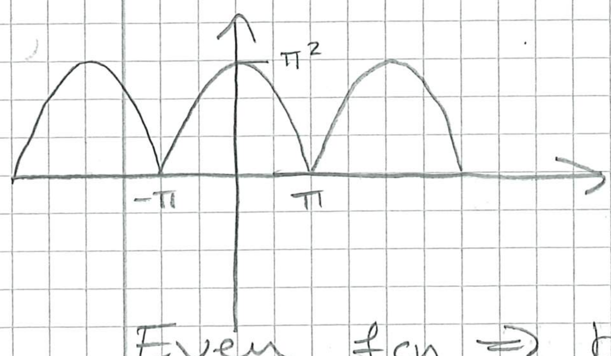
# Solutions to exam 2019-01-09

1a) $f(t) = \pi^2 - t^2 \qquad -\pi \le t \le \pi$




Even fcn $\Rightarrow b_n = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - t^2)\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi}(\pi^2-t^2)\cos(nt)\,dt =$$

$$\underset{n>0}{\overset{\uparrow}{=}} \frac{2}{\pi}\cancel{\left[(\pi^2-t^2)\frac{\sin(nt)}{n}\right]_0^{\pi}}^{\,0} - \frac{2}{\pi}\int_0^{\pi} -2t\,\frac{\sin(nt)}{n}\,dt$$

$$= \frac{4}{\pi n}\int_0^{\pi} t\sin(nt)\,dt = -\frac{4}{\pi n}\left[t\frac{\cos nt}{n}\right]_0^{\pi} +$$

$$+\frac{4}{n\pi}\underbrace{\int_0^{\pi}\frac{\cos nt}{n}\,dt}_{\left[\frac{\sin nt}{n^2}\right]_0^{\pi} = 0} = -\frac{4}{\pi n^2}\pi\cos n\pi = \frac{4}{n^2}(-1)^{n+1}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - t^2)\,dt = \frac{2}{\pi}\left[\pi^2 t - \frac{t^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left(\pi^3 - \frac{\pi^3}{3}\right) =$$

$$= \frac{4\pi^2}{3}$$

So $\quad f(t) \sim \dfrac{2\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos(nt)$

1b) Since $f$ is continuous and the generalised one-sided derivatives exist the Fourier series converges to $f$ everywhere

1c) Take $t=0$. Then

$$\pi^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n$$

$$4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

1d) Parseval: $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\,dt = \frac{|a_0|^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} |a_n|^2 + \underbrace{|b_n|^2}_{=0}$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi^2 - t^2)^2\,dt = \frac{1}{\pi}\int_0^{\pi} \pi^4 - 2t^2\pi^2 + t^4\,dt =$$

$$= \frac{1}{\pi}\left[\pi^4 t - \frac{2}{3}t^3\pi^2 + \frac{1}{5}t^5\right]_0^{\pi} = \frac{1}{\pi}\left(\pi^5 - \frac{2}{3}\pi^5 + \frac{1}{5}\pi^5\right)$$

$$= \pi^4\left(\frac{15}{15} - \frac{10}{15} + \frac{3}{15}\right) = \frac{8}{15}\pi^4$$

Hence $\frac{8}{15}\pi^4 = \frac{4\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{16}{n^4}$

$$\Rightarrow 8\sum_{n=1}^{\infty} \frac{1}{n^4} = \left(\frac{8}{15} - \frac{4}{9}\right)\pi^4$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \left(\frac{1}{15} - \frac{1}{18}\right)\pi^4 = \left(\frac{6}{90} - \frac{5}{90}\right)\pi^4 = \frac{\pi^4}{90}$$

2) $-u''(x) + u(x) = f(x)$

Take the Fourier transform and use F.7:

$\xi^2 \hat{u}(\xi) + \hat{u}(\xi) = \hat{f}(\xi)$

$\Longrightarrow \quad \hat{u}(\xi) = \dfrac{\hat{f}(\xi)}{1+\xi^2}$

Since $\hat{f}(\xi)$ is bounded we have $\dfrac{\hat{f}(\xi)}{1+\xi^2} \in L^1(\mathbb{R})$ and inversion formula yields

$$u(x) = \frac{1}{2\pi}\int \frac{\hat{f}(\xi)}{1+\xi^2} e^{ix\xi} \, d\xi$$

3) Rewrite $f(x) = e^{-x^2} = e^{-(\sqrt{2}x)^2/2}$

Then $\hat{f}(\xi) = \dfrac{1}{\sqrt{2}} \sqrt{2\pi}\, e^{-\left(\frac{\xi}{\sqrt{2}}\right)^2/2} = \sqrt{\pi}\, e^{-\xi^2/4}$

By F.9 $\hat{g}(\xi) = (\hat{f}(\xi))^2 = \pi e^{-\xi^2/2} = \sqrt{\frac{\pi}{2}} \sqrt{2\pi}\, e^{-\xi^2/2}$

So $g(x) = \sqrt{\frac{\pi}{2}}\, e^{-x^2/2}$

4) We use G-S to construct an ON-basis from the vectors $f_0(x) = 1$, $f_1(x) = x$ in the inner product

$\langle f, g \rangle := \int_0^\infty f(x)\overline{g(x)} e^{-x} dx.$

$g_0(x) = f_0(x) = 1, \quad \|g_0\|^2 = \int_0^\infty |g_0(x)|^2 e^{-x} dx =$

$= \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 1$

$e_0(x) = \dfrac{g_0(x)}{\|g_0\|} = 1$

$g_1(x) = f_1(x) - \langle e_0, f_1 \rangle e_0(x)$

We have $\langle e_0, f_1 \rangle = \int_0^\infty x e^{-x}\,dx = \cancel{\left[-xe^{-x}\right]_0^\infty}_{0} + \underbrace{\int_0^\infty e^{-x}\,dx}_{1 \text{ (from before)}} = 1,$

so $g_1(x) = x - 1$. $\|g_1\|^2 = \int_0^\infty (x-1)^2 e^{-x}\,dx =$

$$= \int_0^\infty x^2 e^{-x}\,dx - 2\underbrace{\int_0^\infty x e^{-x}\,dx}_{1 \text{ (from before)}} + \underbrace{\int_0^\infty e^{-x}\,dx}_{1} = \int_0^\infty x^2 e^{-x}\,dx - 1$$

Hence we need to calculate

$$\int_0^\infty x^2 e^{-x}\,dx = \cancel{\left[-x^2 e^{-x}\right]_0^\infty}_{0} + \underbrace{\int_0^\infty 2x e^{-x}\,dx}_{2 \text{ (from before)}} = 2$$

So $\|g_1\|^2 = 1$ and $e_1(x) = g_1(x) = x - 1$.

The coefficients $d_0, d_1$ minimising

$$\int_0^\infty |c_0 e_0(x) + c_1 e_1(x) - \sin x|^2 e^{-x}\,dx$$

are $d_0 = \langle \sin, e_0 \rangle$, $d_1 = \langle \sin, e_1 \rangle$. To calculate these, it is useful to find the primitive function of $e^{-x}\sin x$, so let's do that first.

$$\int e^{-x}\sin x\,dx = -e^{-x}\cos x - \int e^{-x}\cos x\,dx$$

$$= -e^{-x}\cos x - \left(e^{-x}\sin x + \int e^{-x}\sin x\,dx\right)$$

Move this to the other side

$$\Rightarrow 2\int e^{-x}\sin x\,dx = -e^{-x}(\cos x + \sin x)$$

$$\Rightarrow \int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\cos x + \sin x)$$

Now $d_0 = \int_0^\infty \sin x e^{-x} \, dx = -\frac{1}{2}\Big[e^{-x}(\cos x + \sin x)\Big]_0^\infty =$

$= \frac{1}{2}$

$$d_1 = \int_0^\infty (x-1)\sin x e^{-x} \, dx = \int_0^\infty x \sin x e^{-x} \, dx - \underbrace{\int_0^\infty \sin x e^{-x} \, dx}_{1/2}$$

We have using IBP

$$\int_0^\infty x \sin x e^{-x} \, dx = \cancelto{0}{\Big[-\frac{1}{2} x e^{-x}(\sin x + \cos x)\Big]_0^\infty} + \frac{1}{2}\underbrace{\int_0^\infty e^{-x} \sin x \, dx}_{\frac{1}{2}}$$
$$+ \frac{1}{2}\int_0^\infty e^{-x} \cos x \, dx$$

$$\int_0^\infty e^{-x} \cos x \, dx = \frac{1}{2}\int_0^\infty \left(e^{(i-1)x} + e^{-(i+1)x}\right) dx = \frac{1}{2}\left[\frac{e^{(i-1)x}}{i-1} - \frac{e^{-(i+1)x}}{i+1}\right]_0^\infty =$$

$$= -\frac{1}{2}\left(\frac{1}{i-1} - \frac{1}{i+1}\right) = -\frac{1}{2}\,\frac{i+1-(i-1)}{-1-1} = \frac{1}{4}\,2 = \frac{1}{2}$$

Hence $\int_0^\infty x \sin x e^{-x} \, dx = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$

and

$d_1 = \frac{1}{2} - \frac{1}{2} = 0.$

The polynomial minimising the integral

$d_0 e_0(x) + d_1 e_1(x) = \frac{1}{2}\,1 + 0\cdot(x-1) = \frac{1}{2}\cdot 1 + 0\cdot x$, so

$c_1 = \frac{1}{2}$, $c_2 = 0$

5) We homogenise using the ansatz
$u(x,t) = v(x,t) + Ax + B$

BC $\Rightarrow$ $-\pi = v(0,t) + B$, $2\pi = v(\pi,t) + A\pi + B$

This suggests $B = -\pi$, $A\pi = 3\pi \Rightarrow A = 3$

We also obtain $v(x,0) = u(x,0) - Ax - B = \pi - 3x$

Hence

$$\begin{cases} v_t = v_{xx} \\ v(0,t) = v(\pi,t) = 0 \\ v(x,0) = \pi - 3x \end{cases}$$

which is a homogeneous problem.

Now make the ansatz $v(x,t) = X(x)T(t)$.

Then $XT' = X''T$. Divide by $XT$ on both sides so

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

because none of LHS or RHS can be depent on $x$ or $t$

We start with the $X$ equation and consider 3 cases:

Case 1 $\lambda < 0$: $X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$

Here BC $\Rightarrow$ $\begin{cases} 0 = A + B \\ 0 = Ae^{\sqrt{-\lambda}\pi} + Be^{-\sqrt{-\lambda}\pi} \end{cases}$

which has the trivial solution $A = B = 0$.

Case 2 $\lambda = 0$: $X(x) = Ax + B$

BC $\Rightarrow$ $\begin{cases} 0 = A \cdot 0 + B \\ 0 = A\pi + B \end{cases}$ $\Rightarrow$ $A = B = 0$

Case 3 $\lambda>0$: $X(x)=A\sin(\sqrt{\lambda}\,x)+B\cos(\sqrt{\lambda}\,x)$

$X(0)=0 \Rightarrow 0=A\sin 0+B\cos 0 \Rightarrow B=0$

$X(\pi)=0 \Rightarrow 0=A\sin(\sqrt{\lambda}\,\pi)$

$\Rightarrow \sin(\sqrt{\lambda}\,\pi)=0 \Rightarrow \sqrt{\lambda_n}\,\pi=n\pi \quad n=1,2,\dots$

$\Rightarrow \lambda_n=n^2$

Hence $X_n(x)=A_n\sin nx$

Now we consider $T$ equation:

$T_n'(t)=-n^2T_n(t)$

$\Rightarrow T_n(t)=C_ne^{-n^2t}$

The full solution can be written as

$$v(x,t)=\sum_{n=1}^{\infty}X_n(x)T_n(t)=\sum_{n=1}^{\infty}A_n\sin(nx)e^{-n^2t}.$$

Consider the IC: $v(x,0)=\pi-3x$, i.e.

$$\sum_{n=1}^{\infty}A_n\sin(nx)=\pi-3x \qquad x\in(0,\pi)$$

Take $f(x)=\begin{cases}\pi-3x & x\in(0,\pi)\\ -\pi-3x & x\in(-\pi,0]\end{cases}$ (odd extension)

$\{A_n\}$ are the Fourier coefficients of $f$

so $A_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)\,dt \underset{\text{even}}{=} \frac{2}{\pi}\int_0^{\pi}(\pi-3x)\sin(nx)\,dt=$

$$=-\frac{2}{\pi}\left[(\pi-3x)\frac{\cos(nx)}{n}\right]_0^{\pi}+\underbrace{\frac{2}{\pi}\int_0^{\pi}-3\frac{\cos nx}{n}\,dx}_{\left[-3\frac{\sin nx}{n^2}\right]_0^{\pi}=0}=$$

$$=-\frac{2}{\pi}\left((\pi-3\pi)\frac{\cos(n\pi)}{n}-\frac{\pi}{n}\right)=\frac{4(-1)^n+2}{n}$$

Hence $v(x,t)=\sum_{n=1}^{\infty}\frac{4(-1)^n+2}{n}\sin(nx)e^{-n^2t}$

and therefore $u(x,t)=3x-\pi+\sum_{n=1}^{\infty}\frac{4(-1)^n+2}{n}\sin(nx)e^{-n^2t}$

6) Taking Laplace transform gives

$$s^2Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) - 24Y(s) = \frac{28}{s+1}$$

$$Y(s)(s^2+5s-24) - 6s - 30 = \frac{28}{s+1}$$

$$Y(s) = \frac{\frac{28}{s+1} + \frac{6(s+5)(s+1)}{s+1}}{s^2+5s-24}$$

$$\left(\text{obs: } s^2+5s-24=0 \Rightarrow s = -\frac{5}{2} \pm \sqrt{\frac{25}{4}+\frac{96}{4}} = -\frac{5}{2} \pm \sqrt{\frac{121}{4}} = -\frac{5}{2} \pm \frac{11}{2} \Rightarrow s_1 = -8,\ s_2 = 3 \Rightarrow s^2+5s-24 = (s-3)(s+8)\right)$$

$$Y(s) = \frac{28+6(s+5)(s+1)}{(s+1)(s+8)(s-3)} = \frac{28+6s^2+36s+30}{(s+1)(s+8)(s-3)} =$$

$$= \frac{6s^2+36s+58}{(s+1)(s+8)(s-3)} \underset{\text{ansatz}}{=} \frac{A}{s+1} + \frac{B}{s+8} + \frac{C}{s-3}$$

$$6s^2+36s+58 = A(s+8)(s-3) + B(s+1)(s-3) + C(s+1)(s+8)$$

Take $s=-8 \Rightarrow \underbrace{6\cdot 8^2}_{384} - \underbrace{36\cdot 8}_{288} + 58 = B(-7)(-11)$

$B = 2$

Take $s=-1 \Rightarrow \underbrace{6-36+58}_{28} = A(7)(-4)$

$\Rightarrow A = -1$

Take $s=3 \Rightarrow \underbrace{6\cdot 3^2}_{54} + \underbrace{36\cdot 3}_{108} + 58 = C(4)(11)$

$C = 5$

Hence $Y(s) = \frac{5}{s-3} + \frac{2}{s+8} - \frac{1}{s+1}$

Taking inverse Laplace transform gives

$y(x) = 5e^{3x} + 2e^{-8x} - e^{-x}$

7) $T$ is the convolution operator given by

$Tf(x) = g * f(x)$ for $g(x) = \frac{2\sin x}{x}$

Hence $\widehat{Tf}(\xi) = \hat{g}(\xi)\hat{f}(\xi)$

Using F.8 and F.10

$\hat{g}(\xi) = 2\pi \chi_{[-1,1]}(-\xi) \underset{\text{even}}{=} 2\pi \chi_{[-1,1]}(\xi)$

Now Plancherel yields using $|\chi_{[-1,1]}(\xi)| \leq 1$

$\|Tf\|_{L^2}^2 = \frac{1}{2\pi}\|\widehat{Tf}\|_{L^2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |2\pi \chi_{[-1,1]}(\xi)\hat{f}(\xi)|^2 \, d\xi$

$\leq 4\pi^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi = 4\pi^2 \int_{-\infty}^{\infty} |f(x)|^2 \, dx =$

$= 4\pi^2 \|f\|_{L^2}^2$

Thus $\|Tf\|_{L^2} \leq 2\pi \|f\|_{L^2}$

8a) By definition $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$

IBP 2 times (allowed since $f \in C^2(\mathbb{T})$)

yields $\hat{f}(n) = \frac{1}{2\pi}\underbrace{\left[f(t)\frac{e^{-int}}{-in}\right]_{-\pi}^{\pi}}_{\substack{\in C(\mathbb{T}) \text{ and} \\ \text{hence } 0}} + \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(t)\frac{e^{-int}}{in}\,dt$

$$= \frac{1}{2\pi}\underbrace{\left[f'(t)\frac{e^{-int}}{-(in)^2}\right]_{-\pi}^{\pi}}_{\in C(\mathbb{T}) \Rightarrow = 0} + \frac{1}{2\pi}\int_{-\pi}^{\pi} f''(t)\frac{e^{-int}}{(in)^2}\,dt$$

Now triangle inequality gives

$|\hat{f}(n)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f''(t)|\frac{1}{n^2}\,dt \le \frac{C}{n^2}$ where

$$C = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f''(t)|\,dt$$

b) Yes! To prove this, use Weierstrass' M-test. The Fourier series is given by $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$.

But since $|\hat{f}(n)e^{int}| \le \frac{C}{n^2}$ and $\sum_{n\neq 0} \frac{C}{n^2} < \infty$

the M-test applies and the Fourier series is uniformly convergent