Exam with solutions - Fourier analysis

Department of Mathematics
Anders Israelsson
2019-06-14

Exam in Fourier Analysis, 5 credits
1MA211
KandFy, KandMa, Fristående

Writing time: 08:00–13:00. Allowed aids: writing materials, table of formulæ.

There are 8 problems in this exam. For the grades 3, 4 and 5 you should obtain at least 18, 25 and 32 points respectively. You have to motivate every step in your to get the full score from a question.

1. Let f be a 2π periodic function with $f(t) = t$ for $0 \leq t < 2\pi$.
   (a) Find the Fourier series of trigonometric form.
   (b) Is the Fourier series uniformly convergent? Motivate your answer!
   (c) Using the result in a) calculate the series
   $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$
   (d) Calculate the series
   $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

7 points

Solution: **(a)** The Fourier coefficients are calculated as follows: for $n \neq 0$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t\cos(nt)\,\mathrm{d}t = \frac{1}{\pi}\left[t\frac{\sin nt}{n}\right]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi}\frac{\sin nt}{n}\,\mathrm{d}t = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t\sin(nt)\,\mathrm{d}t = -\frac{1}{\pi}\left[t\frac{\cos nt}{n}\right]_0^{2\pi} + \cancel{\frac{1}{\pi}\int_0^{2\pi}\frac{\cos nt}{n}}\,\mathrm{d}t$$
$$= -2\frac{\cos(2n\pi)}{n} = -\frac{2}{n}$$

and for $n = 0$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t\,\mathrm{d}t = \frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^{2\pi} = 2\pi$$

Thus the Fourier series is given by

$$f(t) \sim \pi - 2\sum_{n=1}^{\infty}\frac{1}{n}\sin(nt)$$

(b) The partial sums of the Fourier series are continuous, but the limit function is not in the points $2\pi n$, $n \in \mathbb{Z}$. This contradicts the fact that the limit of every uniformly convergent sequence of continuous functions is again continuous and hence the answer is **no**.

(c) Insert $t = \frac{\pi}{2}$ into the result obtained from (a). Then observe that

$$\sin(n\frac{\pi}{2}) = \begin{cases} (-1)^{\frac{n-1}{2}}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

so, setting $n = 2k+1$, the Fourier series simplify to

$$\pi - 2\sum_{k=1}^{\infty} \frac{1}{2k+1}(-1)^k$$

Since f is continuous in $t = \frac{\pi}{2}$ Dirichlet's theorem states that the value of the Fourier series in that point is $\frac{\pi}{2}$, i.e.

$$\pi - 2\sum_{k=1}^{\infty} \frac{1}{2k+1}(-1)^k = \frac{\pi}{2}$$

or

$$\sum_{k=1}^{\infty} \frac{1}{2k+1}(-1)^k = \frac{\pi}{4}$$

(d) We want to use Parseval's theorem. To this end we calculate

$$\int_0^{2\pi} |t|^2 \, \mathrm{d}t = \left[\frac{t^3}{3}\right]_0^{2\pi} = \frac{8\pi^3}{3}$$

Thus Parseval gives

$$\frac{1}{2\pi}\frac{8\pi^3}{3} = \frac{(2\pi)^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(-\frac{2}{n}\right)^2$$

This simplifies to

$$\frac{4\pi^2}{3} = \pi^2 + 2\sum_{n=1}^{\infty}\frac{1}{n^2} \Leftrightarrow$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

2. Calculate the Fourier transform of $f(x) = x\chi_{[-1,1]}(x)$, using the definition.

5 points

Solution: By definition

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}} e^{-ix\xi} x\chi_{[-1,1]}(x)\,\mathrm{d}x = \int_{-1}^{1} xe^{-ix\xi}\,\mathrm{d}x \\ &\underset{\text{IBP}}{=} \left[x\frac{e^{-ix\xi}}{-i\xi}\right]_{-1}^{1} - \int_{-1}^{1}\frac{e^{-ix\xi}}{-i\xi}\,\mathrm{d}x = \left(\frac{e^{-i\xi}+e^{i\xi}}{-i\xi}\right) - \left[\frac{e^{-ix\xi}}{-\xi^2}\right]_{-1}^{1} \\ &= \frac{1}{\pi}\frac{\cos(\xi)}{-i\xi} + \frac{e^{-i\xi}-e^{i\xi}}{\xi^2} = 2i\frac{\cos(\xi)}{\xi} - 2i\frac{\sin(\xi)}{\xi^2}\end{aligned}$$

3. Calculate the integral

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$$

Hint: What is the Fourier transform of $\frac{\sin x}{x}$? You may assume that all integrals converge.

5 points

Solution: One can consider the integral as half of the Fourier transform of $\frac{\sin x}{x}$ in the point 0, i.e.

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty e^{ix0}\frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{2}\widehat{\left(\frac{\sin x}{x}\right)}(0)$$

By the table of formulæ

$$\chi_{[-1,1]}(x) \overset{\mathcal{F}}{\rightsquigarrow} \frac{2\sin\xi}{\xi}$$

$$\Rightarrow \frac{2\sin x}{x} \overset{\mathcal{F}}{\rightsquigarrow} 2\pi\chi_{[-1,1]}(-\xi) = 2\pi\chi_{[-1,1]}(\xi)$$

$$\Rightarrow \frac{\sin x}{x} \overset{\mathcal{F}}{\rightsquigarrow} \pi\chi_{[-1,1]}(\xi)$$

Therefore $\widehat{\left(\frac{\sin x}{x}\right)}(0) = \pi\chi_{[-1,1]}(0) = \pi$ and

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

4. Determine a complete orthogonal system in $L^2([0,\pi])$ consisting of solutions of the problem

$$\begin{cases} u''(x) + \lambda u(x) = 0, & 0 < x < \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

4 points

Solution: We have to consider 3 different cases:

Case 1: $\lambda < 0$
We obtain the solution

$$\begin{cases} u(x) = C_1 e^{i\sqrt{-\lambda}x} + C_2 e^{-i\sqrt{-\lambda}x}, & 0 \le x \le \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

$$u(0) = 0 \Rightarrow C_1 + C_2 = 0 \Rightarrow u(x) = C_1\left(e^{i\sqrt{-\lambda}x} + e^{-i\sqrt{-\lambda}x}\right)$$

$$u(\pi) = 0 \Rightarrow C_1 = 0$$

Case 2: $\lambda = 0$
We obtain the solution

$$\begin{cases} u(x) = C_1 x + C_2, & 0 \le x \le \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

$$u(0) = 0 \Rightarrow C_2 = 0 \Rightarrow u(x) = C_1 x$$

$$u(\pi) = 0 \Rightarrow C_1 = 0$$

Case 3: $\lambda > 0$
We obtain the solution

$$\begin{cases} u(x) = C_1 \cos\left(\sqrt{\lambda}x\right) + C_2 \sin\left(\sqrt{\lambda}x\right), & 0 \le x \le \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

$$\begin{aligned} u(0) = 0 \Rightarrow C_1 = 0 \Rightarrow u(x) = C_2 \sin\left(\sqrt{\lambda}x\right) \\ u(\pi) = 0 \Rightarrow \sin\left(\sqrt{\lambda}\pi\right) = 0 \\ \Rightarrow \sqrt{\lambda_n}\pi = n\pi, \quad n = 1, 2, \ldots \\ \Rightarrow \lambda_n = n^2 \end{aligned}$$

A set of eigenfunctions to the problem will be

$$u_n(x) = \sin(nx)$$

By Sturm-Liouville's theorem the set $\{u_n\}_{n=1}^{\infty}$ with eigenvalues $\lambda_n = n^2$ is a complete orthogonal system in $L^2([0, \pi])$.

5. For the Fourier coefficients a_n, b_n and c_n defined as in the table of formulæ, show that $a_n = c_n + c_{-n}$ and $b_n = i(c_n - c_{-n})$.

4 points

Solution: From the definition of the Fourier coefficients

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{\mathbb{T}} f(t) \cos(nt)\, \mathrm{d}t = \frac{1}{\pi} \int_{\mathbb{T}} f(t) \frac{e^{it} + e^{-it}}{2}\, \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{it}\, \mathrm{d}t + \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-it}\, \mathrm{d}t = c_{-n} + c_n \\ b_n &= \frac{1}{\pi} \int_{\mathbb{T}} f(t) \sin(nt)\, \mathrm{d}t = \frac{1}{\pi} \int_{\mathbb{T}} f(t) \frac{e^{it} - e^{-it}}{2i}\, \mathrm{d}t \\ &= -i \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{it}\, \mathrm{d}t + i \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-it}\, \mathrm{d}t = i(-c_{-n} + c_n) \end{aligned}$$

6. Find all bounded functions f with $D_f = [0, \infty)$ that solve the integral equation

$$\int_0^t f(t - s) \cos(4s)\, \mathrm{d}s = t \sin(4t).$$

5 points

Solution: By taking the Laplace transform we have

$$\begin{aligned} F(s) \frac{s}{s^2 + 4^2} &= \frac{2 * 4s}{(s^2 + 4^2)^2} \\ \Rightarrow F(s) &= 2 \frac{4}{s^2 + 4^2} \\ \Rightarrow f(t) &= 2 \sin(4t) \end{aligned}$$

7. (a) Let A be a symmetric operator in an inner product space with distinct eigenvalues μ and λ (i.e. $\mu \neq \lambda$). Show that two eigenvectors corresponding to μ and λ respectively must be orthogonal. Hint: Eigenvalues of a symmetric operator are always real-valued.

(b) Show that the set of functions $\{e^{inx} \mid n \in \mathbb{Z}\}$ is an orthonormal set with respect to the inner product space $L^2(\mathbb{T}, w)$ with weight $w = \frac{1}{2\pi}$.

5 points

Solution: **(a)** Let u and v be eigenvectors corresponding to the eigenvalues μ and λ.

$$\mu\langle u, v\rangle = \langle \mu u, v\rangle = \langle Au, v\rangle = \langle u, Av\rangle = \langle u, \lambda v\rangle = \lambda\langle u, v\rangle$$

Since $\mu \neq \lambda$, the only possibility is that $\langle u, v\rangle = 0$ which is the definition of orthogonality.

(b) Note that the inner product is given by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\overline{g(x)}\,\mathrm{d}x$$

Pairing two functions in the given set yields

$$\langle e^{imx}, e^{inx}\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} e^{imx}\overline{e^{inx}}\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)x}\,\mathrm{d}x = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

8. Solve the PDE given by

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + 2\cos\left(\frac{9}{2}x\right), & 0 < x < \pi, t > 0 \\ u_x(0,t) = 0, u(\pi,t) = 0, & t > 0 \\ u(x,0) = 1 + \frac{8}{81}\cos\left(\frac{9}{2}x\right), & 0 < x < \pi \end{cases}$$

5 points

Solution: First we need to homogenise the problem, since we have an inhomogeneuos term in the equation. We want

$$\begin{cases} v_t(x,t) = v_{xx}(x,t), & 0 < x < \pi, t > 0, \\ v_x(0,t) = v(\pi,t) = 0, & t > 0. \end{cases}$$

Make the ansatz $v(x,t) = u(x,t) + Ax + B + C\sin\left(\frac{9}{2}x\right) + D\cos\left(\frac{9}{2}x\right)$. Then $v_t(x,t) = v_{xx}(x,t)$ and BC yields $u_t(x,t) = u_{xx}(x,t) - \frac{81}{4}C\sin\left(\frac{9}{2}x\right) - \frac{81}{4}D\cos\left(\frac{9}{2}x\right)$, $v_x(0,t) = A + \frac{9}{2}C$, $v(\pi,t) = A\pi + B + C\sin\left(\frac{9}{2}\pi\right) + D\cos\left(\frac{9}{2}\pi\right)$, so we obtain

$$\begin{cases} C = 0 \\ -\frac{81}{4}D = 2 \\ A + \frac{9}{2}C = 0 \\ A\pi + B - C = 0 \end{cases}$$

This has the solution

$$\begin{cases} A = 0 \\ B = 0 \\ C = 0 \\ D = -\frac{8}{81} \end{cases}$$

and hence $v(x,t) = u(x,t) - \frac{8}{81}\cos\left(\frac{9}{2}x\right)$. Now the original problem has turned into

$$\begin{cases} v_t(x,t) = v_{xx}(x,t), & 0 < x < \pi, t > 0, \\ v_x(0,t) = v(\pi,t) = 0, & t > 0, \\ v(x,0) = 1, & 0 < x < \pi \end{cases}$$

Make the assumption $v(x,t) = X(x)T(t)$. Then $X(x)T'(t) = X''(x)T(t)$ or

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

We consider the X-equation. Divide into three cases.

Case 1 - $\lambda < 0$
The solution is given by

$$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$$

Now BC yields $X'(0) = X(\pi) = 0$ so

$$\begin{cases} C_1\sqrt{-\lambda} - C_2\sqrt{-\lambda} = 0 \\ C_1 e^{\sqrt{-\lambda}\pi} + C_2 e^{-\sqrt{-\lambda}\pi} = 0 \end{cases}$$

The system of equation corresponds to matrix with determinant $\neq 0$ so the unique solution is $C_1 = C_2 = 0$.

Case 2 - $\lambda = 0$
We have $X(x) = C_1 x + C_2$. $X'(0) = X(\pi) = 0 \Rightarrow C_1 = C_2 = 0$.

Case 3 - $\lambda > 0$
The solution is given by

$$X(x) = C_1\cos\left(\sqrt{\lambda}x\right) + C_2\sin\left(\sqrt{\lambda}x\right)$$

$X'(0) = 0 \Rightarrow -C_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}0\right) + C_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}0\right) = 0 \Rightarrow C_2 = 0$. $X(\pi) = 0$ yields

$$C_1\cos\left(\sqrt{\lambda}\pi\right) = 0$$

which has the solution

$$\sqrt{\lambda_n}\pi = \left(n + \frac{1}{2}\right)\pi, \quad n = 0, 1, 2, \ldots$$

or

$$\lambda_n = \left(n + \frac{1}{2}\right)^2$$

Case 3 yields

$$X_n(x) = C_n\cos\left(\left(n + \frac{1}{2}\right)x\right), \quad n = 0, 1, 2, \ldots$$

Turning to T we want to find the solution to the equation

$$T_n'(t) = -\left(n + \frac{1}{2}\right)^2 T_n(t)$$

which is

$$T_n(t) = D_n e^{-\left(n+\frac{1}{2}\right)^2 t}$$

Hence the general solution will be given

$$v(x,t) = \sum_{n=0}^{\infty} X_n(x) T_n(t) = \sum_{n=0}^{\infty} D_n e^{-\left(n+\frac{1}{2}\right)^2 t} \cos\left(\left(n + \frac{1}{2}\right) x\right)$$

By the first IC (of v)

$$1 = v(x,0) = \sum_{n=0}^{\infty} D_n \cos\left(\left(n + \frac{1}{2}\right) x\right)$$

We want to find an expression for D_n and by the theory of Fourier series we know that this exists and is unique. To find this expression multiply by $\cos\left(\left(m + \frac{1}{2}\right) x\right)$ for any integer $m \geq 0$ and integrate over $0 < x < \pi$:

$$\begin{aligned}
\int_0^\pi \cos\left(\left(m + \frac{1}{2}\right) x\right) \mathrm{d}x &= \int_0^\pi \cos\left(\left(m + \frac{1}{2}\right) x\right) \sum_{n=0}^{\infty} D_n \cos\left(\left(n + \frac{1}{2}\right) x\right) \mathrm{d}x \\
\left[\frac{\sin\left(\left(m+\frac{1}{2}\right)x\right)}{m+\frac{1}{2}}\right]_0^\pi &= \sum_{n=0}^{\infty} D_n \int_0^\pi \cos\left(\left(m + \frac{1}{2}\right) x\right) \cos\left(\left(n + \frac{1}{2}\right) x\right) \mathrm{d}x \\
\frac{\sin\left(\left(m+\frac{1}{2}\right)\pi\right)}{m+\frac{1}{2}} &= \sum_{n=0}^{\infty} \frac{1}{2} D_n \int_0^\pi \left(\cos\left((m-n)x\right) + \cancel{\cos\left((m+n+1)x\right)}\right) \mathrm{d}x
\end{aligned}$$

We have

$$\int_0^\pi \cos\left((m-n)x\right) \mathrm{d}x = \begin{cases} \pi, & m = n \\ 0, & m \neq n \end{cases}$$

Thus only one term in the sum survives ($n = m$), and

$$\frac{(-1)^m}{m+\frac{1}{2}} = \frac{1}{2} D_m \pi$$

or

$$D_n = \frac{4(-1)^n}{\pi(2n+1)}$$

Hence

$$v(x,t) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi(2n+1)} e^{-\left(n+\frac{1}{2}\right)^2 t} \cos\left(\left(n + \frac{1}{2}\right) x\right)$$

and

$$u(x,t) = \frac{8}{81} \cos\left(\frac{9}{2} x\right) + \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi(2n+1)} e^{-\left(n+\frac{1}{2}\right)^2 t} \cos\left(\left(n + \frac{1}{2}\right) x\right)$$