Homework exam, Integration theory, November 2018

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Your answers (individually composed) should be submitted no later than 24.00 on Monday 12 November. For 3 bonus points on the final exam you should have 4 correct solution, for 2 bonus points you should have 3 correct solutions, and for 1 bonus point you should have at least 2 correct solutions. The Lebesgue measure is denoted by m.

1. Suppose that $\{E_j\}_{j=1}^{\infty}$ is a sequence of pairwise disjoint Lebesgue measurable sets with $0 < m(E_j) < \infty$. Show that there is a measurable function f, integrable on each set E_j, and so that

$$\sum_{j=1}^{\infty} \int_{E_j} f(x)\, dx$$

is convergent, but so that f is not integrable on $\bigcup_{j=1}^{\infty} E_j$.

2. Compute

$$\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^{-n} \left(1-\sin\frac{x}{n}\right)\, dx.$$

3. On a measure space $(X, \mathcal{M}, \mu)$, suppose that $\int |f_n - f|\, d\mu \to 0$ and $f_n \to g$ a.e., as $n \to \infty$. Prove that $f = g$ a.e.

4. Prove that a function f is integrable on a finite measure space (X, μ) if and only if

$$\sum_{k=1}^{\infty} \mu(\{x \in X : |f(x)| \geq k\}) < \infty.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$, if $x \in (-1, 1]$, and define for all other $x \in \mathbb{R}$ f so that it becomes a periodic function of period 2. Let $\{\lambda_n\}_{n=1}^{\infty}$ be an arbitrary sequence of real numbers, and let $\{k_n\}_{n=1}^{\infty}$ be a sequence of positive numbers that satisfies

$$\sum_{n=1}^{\infty} \frac{1}{k_n} < \infty.$$

Prove that

$$\sum_{n=1}^{\infty} f(\lambda_n x)^{k_n}$$

converges for Lebesgue almost every x in $\mathbb{R}$.

6. Show that the limit

$$\lim_{n\to\infty} \log n \int_0^{\infty} \frac{(1+y)^{-n}}{y(\log^2 y + \pi^2)}\, dy$$

exists, and find the limit. (*Hint:* make the substitution $y = n^t$.)
[This integral, and the limit, is an authentic example from the research this semester of one of the teachers.]