# SOLUTIONS TO HOMEWORK EXAMINATION, INTEGRATION THEORY, NOVEMBER 2018

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**1.** *Remark.* This is impossible with $f_n \geqslant 0$; we need cancellation. This can be achieved in (at least) two different ways. Note also that the result is valid for any measure $\mu$.

*First solution.*
Define $f$ by

$$f(x) = \frac{(-1)^i}{im(E_i)}, \qquad x \in E_i.$$

Then $f$ is integrable on each $E_i$, with $\int_{E_i} f \, \mathrm{d}m = (-1)^i/i$, and thus the sum $\sum_i \int_{E_i} f \, \mathrm{d}m$ converges. However,

$$\int_{\bigcup_i E_i} |f| \, \mathrm{d}m = \sum_{i=1}^{\infty} \int_{E_i} |f| \, \mathrm{d}m = \sum_{i=1}^{\infty} \frac{1}{i} = \infty,$$

and thus $f$ is not integrable on $\bigcup_i E_i$.

*Second solution.*
Partition each $E_i$ as $E_i = E_i' \cup E_i''$, with $E_i' \cap E_i'' = \emptyset$ and $m(E_i') = m(E_i'') = m(E)/2$. (This is always possible with Lebesgue measure, as was assumed; it is not always possible if, e.g., we instead have a discrete measure.)

Define $f$ by

$$f(x) := \begin{cases} 1/m(E_i), & x \in E_i', \\ -1/m(E_i), & x \in E_i''. \end{cases}$$

Then $f$ is integrable on each $E_i$, with $\int_{E_i} f \, \mathrm{d}m = 0$, and thus the sum $\sum_i \int_{E_i} f \, \mathrm{d}m$ converges trivially. However,

$$\int_{\bigcup_i E_i} |f| \, \mathrm{d}m = \sum_{i=1}^{\infty} \int_{E_i} |f| \, \mathrm{d}m = \sum_{i=1}^{\infty} 1 = \infty,$$

and thus $f$ is not integrable on $\bigcup_i E_i$.

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**2.** We have

$$\int_0^n \Big(1+\frac{x}{n}\Big)^{-n}\Big(1-\sin\frac{x}{n}\Big)\,\mathrm{d}x = \int_0^\infty \chi_{(0,n)}(x)\Big(1+\frac{x}{n}\Big)^{-n}\Big(1-\sin\frac{x}{n}\Big)\,\mathrm{d}x$$
$$= \int_0^\infty f_n(x)\,\mathrm{d}x, \tag{2.1}$$

say. For any fixed $x>0$, we have as $n\to\infty$, $\chi_{(0,n)}(x)\to 1$, $\big(1+x/n\big)^{-n}\to e^{-x}$ and $1-\sin\frac{x}{n}\to 1$, and thus $f_n(x)\to e^{-x}$.

We want to apply dominated convergence[1], and we thus seek a dominating function. We give two alternative ways to do so. (Of course, one is enough.)

*First domination.* By the binomial expansion, for any $n\geqslant 2$ and $x\geqslant 0$,

$$\Big(1+\frac{x}{n}\Big)^n = \sum_{k=0}^n \binom{n}{k}\frac{x^n}{n^n} \geqslant 1+x+\frac{n(n-1)}{2}\cdot\frac{x^2}{2} \geqslant 1+\frac{x^2}{4}.$$

Hence, for $n\geqslant 2$,

$$\Big(1+\frac{x}{n}\Big)^{-n} \leqslant \frac{1}{1+x^2/4},$$

which is integrable on $(0,\infty)$.

This yields a domination $f_n(x)\leqslant g(x) := 2\big(1+x^2/4\big)^{-1}$ for all $x\in(0,\infty)$, and all $n\geqslant 2$. This suffices, since it suffices to consider $n\geqslant 2$ when we take limits.[2]

*Second (alternative) domination.* By the Taylor series expansion, we have for $0\leqslant x\leqslant 1$,

$$\log(1+x) = \sum_{k=1}^\infty (-1)^k\frac{x^k}{k} = \Big(x-\frac{x^2}{2}\Big)+\Big(\frac{x^3}{3}-\frac{x^4}{4}\Big)+\ldots \geqslant x-\frac{x^2}{2} \geqslant \frac{1}{2}x.$$

Hence, for $0\leqslant x\leqslant n$,

$$\big(1+\frac{x}{n}\big)^{-n} = e^{-n\log(1+x/n)} \leqslant e^{-n\cdot x/2n} = e^{-x/2},$$

and thus

$$|f_n(x)| \leqslant 2e^{-x/2}. \tag{2.2}$$

Furthermore, (2.2) is trival for $x>n$ because then $f_n(x)=0$; hence (2.2) holds for all $x$ and $n$. The function $h(x) := 2e^{-x/2}$ is thus a dominating function, and it is integrable on $[0,\infty)$.

*Conclusion.* By any of the two dominations above (or another one), the theorem of dominate convergence applies to (2.1), and yields

$$\int_0^n \Big(1+\frac{x}{n}\Big)^{-n}\Big(1-\sin\frac{x}{n}\Big)\,\mathrm{d}x = \int_0^\infty f_n(x) \to \int_0^\infty e^{-x}\,\mathrm{d}x = 1.$$

[1] Monotone convergence is not applicable in this problem

[2] In fact, $1+x\geqslant 1+x^2/4$ for $x\in(0,1)$, so this domination happens to hold for $n=1$ too. But it is simpler to ignore $n=1$ instead of finding a special argument for that case.

**3.** *First solution.*
Fatou's lemma yields, using the two assumptions,

$$\int_X |f-g|\,\mathrm{d}\mu = \int_X \liminf_{n\to\infty} |f-f_n|\,\mathrm{d}\mu \leqslant \liminf_{n\to\infty} \int_X |f-f_n|\,\mathrm{d}\mu = 0.$$

Hence, $f = g$ a.e.

*Second solution.*
Let $A$ be a subset of $X$ with finite measure. $\int |f_n - f|\,\mathrm{d}\mu \to 0$ implies that $f_n \to f$ in measure. $f_n \to g$ a.e. implies that $f_n \to g$ in measure on $A$. Thus $f_n$ converges to both $f$ and $g$ in measure on $A$. Since a limit in measure is a.e. unique, this shows $f = g$ a.e. on $A$. This holds for every $A$ of finite measure, and it follows that $f = g$ a.e. on $X$. (The last step is easy if $X$ is $\sigma$-finite. In general it requires an extra argument using that $B_n := \{x : |f_n(x) - f(x)| > 0\}$ is $\sigma$-finite, and thus $B := \bigcup_n B_n$ is $\sigma$-finite, and that on $X \setminus B$ $f_n = f$ for all $n$ and thus $g = f$ a.e. We omit the details.)

*Third solution.*
$\int |f_n - f|\,\mathrm{d}\mu \to 0$ implies that there exists a subsequence $n_k$ such that $f_{n_k} \to f$ a.e. Hence, for a.e. $x$, the full sequence $f_n(x) \to g(x)$, and the subsequence $f_{n_k}(x) \to f(x)$; consequently, $g(x) = f(x)$.

**4.** Let $E_k := \{x \in X : |f(x)| \geqslant k\}$. Then, using Beppo Levi's theorem,

$$\sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{\infty} \int_X \chi_{E_k}(x)\,\mathrm{d}\mu(x) = \int_X \sum_{k=1}^{\infty} \chi_{E_k}(x)\,\mathrm{d}\mu(x). \tag{4.1}$$

Furthermore, for every $x \in X$,

$$\sum_{k=1}^{\infty} \chi_{E_k}(x) = \sum_{1\leqslant k\leqslant |f(x)|} 1 = \lfloor |f(x)| \rfloor,$$

and thus

$$\sum_{k=1}^{\infty} \chi_{E_k}(x) \leqslant |f(x)| \leqslant 1 + \sum_{k=1}^{\infty} \chi_{E_k}(x).$$

Integrating over $x$ yields

$$\int_X \sum_{k=1}^{\infty} \chi_{E_k}(x)\,\mathrm{d}\mu(x) \leqslant \int_X |f(x)|\,\mathrm{d}\mu(x) \leqslant \mu(X) + \int_X \sum_{k=1}^{\infty} \chi_{E_k}(x)\,\mathrm{d}\mu(x), \tag{4.2}$$

and the result follows by (4.1) and (4.2). (Recall that $\mu(X) < \infty$.)

**5.** Let $A > 0$. We will show that

$$\int_0^A \sum_{n=1}^{\infty} |f(\lambda_n x)^{k_n}|dx < \infty.$$

This implies that the sum is finite a.e. on $[0, A]$. Since $A$ is arbitrary, and the sum is an even function, this implies that the sum is finite a.e. on $\mathbb{R}$, To show that this integral is finite, we first use Beppo Levi:

$$\int_0^A \sum_{n=1}^{\infty} |f(\lambda_n x)^{k_n}|dx = \sum_{n=1}^{\infty} \int_0^A |f(\lambda_n x)^{k_n}|dx.$$

To estimate the integrals $\int_0^A |f(\lambda_n x)^{k_n}|dx$, I consider two cases. [3] We may for convenience assume that $\lambda_n \geqslant 0$, since replacing $\lambda_n$ by $|\lambda_n|$ does not change the integral.

**Case 1:** $\lambda_n A \leqslant 1$. Then $f(\lambda_n x) = \lambda_n x$ for $x \in [0, A]$, and thus

$$\int_0^A |f(\lambda_n x)^{k_n}|dx = \int_0^A (\lambda_n x)^{k_n} dx = \frac{\lambda_n^{k_n} A^{k_n+1}}{k_n+1} = \frac{(\lambda_n A)^{k_n} A}{k_n+1} \leqslant \frac{A}{k_n+1}.$$

**Case 2:** $\lambda_n A > 1$. Let $N$ be the smallest integer $\geqslant \lambda_n A$. Note that, since $\lambda_n A > 1$, $N \leqslant 2\lambda_n A$. Then, using the symmetry and periodicity of $f$,

$$\begin{aligned}\int_0^A |f(\lambda_n x)^{k_n}|dx &\leqslant \int_0^{N/\lambda_n} |f(\lambda_n x)^{k_n}|dx = N \int_0^{1/\lambda_n} |f(\lambda_n x)^{k_n}|dx \\ &= N \int_0^{1/\lambda_n} (\lambda_n x)^{k_n} dx = N \frac{1/\lambda_n}{k_n+1} \leqslant \frac{2A}{k_n+1}.\end{aligned}$$

Hence, in both cases the integral is at most

$$\frac{2A}{k_n+1} \leqslant \frac{2A}{k_n}.$$

Consequently,

$$\sum_{n=1}^{\infty} \int_0^A |f(\lambda_n x)^{k_n}|dx \leqslant \sum_{n=1}^{\infty} \frac{2A}{k_n} = 2A \sum_{n=1}^{\infty} \frac{1}{k_n} < \infty.$$

This completes the proof.

**6.** With the substitution $y = n^t = e^{t \log n}$ in the hint, we obtain, for $n \geqslant 1$,

$$\log n \int_0^{\infty} \frac{(1+y)^{-n}}{y(\log^2 y + \pi^2)} \,\mathrm{d}y = \int_{-\infty}^{\infty} \log^2 n \frac{(1+n^t)^{-n}}{t^2 \log^2 n + \pi^2} \,\mathrm{d}t =: \int_{-\infty}^{\infty} f_n(t) \,\mathrm{d}t, \tag{6.1}$$

say. We have

$$f_n(t) = \log^2 n \frac{(1+n^t)^{-n}}{t^2 \log^2 n + \pi^2} = \frac{\log^2 n}{t^2 \log^2 n + \pi^2} (1+n^t)^{-n}. \tag{6.2}$$

We first show that $f_n(t)$ converges a.e. as $n \to \infty$. Consider the two factors separately. We have, as $n \to \infty$ with $t$ fixed,

$$\frac{\log^2 n}{t^2 \log^2 n + \pi^2} \to \frac{1}{t^2}, \qquad t \neq 0. \tag{6.3}$$

Furthermore, if $t \geqslant 0$, then $(1+n^t)^{-n} \leqslant 2^{-n} \to 0$; if $-\infty < t < 0$, then (for fixed $t$) $n^t \to 0$ and thus

$$\log\bigl((1+n^t)^{-n}\bigr) = -n \log(1+n^t) = -n\bigl(n^t + O(n^{2t})\bigr)$$

[3] This is not necessary, but I find it simplest this way.

$$= -n^{t+1}\bigl(1+o(1)\bigr) \to \begin{cases} 0, & t<-1, \\ -\infty, & -1<t<0. \end{cases}$$

Hence,

$$(1+n^t)^{-n} \to \begin{cases} 1, & t<-1, \\ 0, & t>-1,\, t\neq 0. \end{cases} \tag{6.4}$$

Consequently, as $n\to\infty$, by (6.3) nd (6.4),

$$f_n(t) \to \begin{cases} 1/t^2, & t<-1, \\ 0, & t>-1,\, t\neq 0. \end{cases} \tag{6.5}$$

Next, we try to dominate $f_n(t)$. We do this in two different ways. First, for any $t\neq 0$,

$$|f_n(t)| \leqslant \frac{\log^2 n}{t^2\log^2 n+\pi^2} \leqslant \frac{1}{t^2}. \tag{6.6}$$

On the other hand, if $|t|\leqslant \frac12$, then $n^t \geqslant n^{-1/2}$ and thus, for $n\geqslant 4$,

$$\log\bigl(1+n^t\bigr) \geqslant \log\bigl(1+n^{-1/2}\bigr) \geqslant \tfrac12 n^{-1/2}. \tag{6.7}$$

Hence,

$$|f_n(t)| \leqslant \frac{\log^2 n}{\pi^2}(1+n^t)^{-n} \leqslant \log^2 n e^{-\frac12 n^{1/2}}. \tag{6.8}$$

The right-hand side is independent of $t$, and $\to 0$ as $n\to\infty$, and is thus bounded by some constant $C$ for all $n\geqslant 4$.

Combining this with (6.6), we find that for every $n\geqslant 4$ and a.e. $t\in\mathbb{R}$,

$$|f_n(t)| \leqslant \min\Bigl(\frac{1}{t^2}, C\Bigr) =: g(t). \tag{6.9}$$

The function $g(t)$ defined in (6.9) is integrable, and thus dominated convergence applies and yields, using (6.5),

$$\int_{-\infty}^{\infty} f_n(t)\,\mathrm{d}t \to \int_{-\infty}^{\infty} \frac{\mathbf{1}\{t<-1\}}{t^2}\,\mathrm{d}t = \int_{-\infty}^{-1} \frac{1}{t^2}\,\mathrm{d}t = 1. \tag{6.10}$$

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