UPPSALA UNIVERSITET
Matematiska Institutionen
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PROV I MATEMATIK
Integration Theory, 1MA215
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Grades 3, 4, 5 normally requires a minimum of 18, 25, 32 credits.

1. Define μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by letting $\mu(A)$ be the number of rational numbers in A (of course $\mu(A) = +\infty$ if there are infinitely many rational numbers in A). Show that μ is a σ-finite measure. (5p)

2. On a measure space with measure μ we have measurable sets $\{A_n\}_{n\geq 1}$, A, and B. Recall that a sequence of measurable functions $\{f_n\}$ converges to f in μ-measure, if for all $\varepsilon > 0$, $\mu(|f_n - f| > \varepsilon) \to 0$, $n \to \infty$. Assume first that the sequence of indicator functions $\{1_{A_n}\}$ converges *μ-almost everywhere* to 1_A *and* converges in *μ-measure* to 1_B. Prove that $\mu(B) = \mu(A)$. Secondly, assume that all we know is that $\{1_{A_n}\}$ converges in *μ-measure* to 1_B. Prove that $\mu(B) = \mu(\liminf_n A_n)$. (6)

3. Let $(X, \mathcal{A}, \mu)$ be a measure space and let f and g be two non-negative functions in $L^1(X, \mu)$. Define the sequence $\{h_n\}$ by $h_n = (f^n + g^n)^{1/n}$, $n \geq 1$. Show that h_n belongs to $L^1(X, \mu)$ for each $n \geq 1$ and that
$$\lim_{n\to\infty} \int_X h_n \, d\mu = \frac{1}{2} \int_X (f + g + |f - g|) \, d\mu. \qquad \text{(6p)}$$

4. Let μ be a σ-finite measure on the real line, which is singular with respect to Lebesgue measure, $\mu \perp m$. Let A be a set with Lebesgue measure $m(A) = 0$. Show that almost all translates of A are zero sets under μ, that is, $\mu(A + x) = 0$, m-a.e. x. (In spite of the fact that there must exist some set A with $m(A) = 0$ and $\mu(A) > 0$, since μ is singular!) (6)

5. Suppose that f is a real-valued function on $[0, 1]$ such that the function $f(x)/x^2$ belongs to $L^2((0, 1), dx)$, and that $\{a_n\}$ is a sequence in $\ell_{3/2}$, that is, $\sum_n |a_n|^{3/2} < \infty$. Prove that the sum $\sum_{n=1}^{\infty} f(a_n x)$ converges almost everywhere on $[0, 1]$. (5p)

6. Show that a function f on the real interval $[0, 1]$ satisfies $|f(x) - f(y)| \leq M|x - y|$ for some $M < \infty$ and all $x, y \in [0, 1]$ if and only if there is a bounded measurable function g such that $f(x) = f(0) + \int_0^x g(y) \, dy$. (6p)

7. Suppose f is an absolutely continuous real-valued function on $\mathbb{R}$ with derivative $f' \in L^p(\mathbb{R})$, $1 \leq p < \infty$.

    a) Prove that for each $x \in \mathbb{R}$, $h^{1-p}(f(x + h) - f(x))^p \to 0 \qquad h \to 0.$ (3p)

    b) Under the additional assumption that $f(x) = 0$, $x \notin [0, 1]$, and we also have $f(0) = 0$, prove that
    $$\int_0^1 |f(x)|^p \, dx \leq \frac{1}{p} \int_0^1 |f'(x)|^p \, dx \qquad \text{(3p)}$$