UPPSALA UNIVERSITET
Matematiska Institutionen
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PROV I MATEMATIK
Integration Theory, 1MA215
2022-06-08

Grades 3, 4, 5 normally requires a minimum of 18, 25, 32 credits.

1. a) Let $E \subset \mathbb{R}$. Show that $\mathcal{F} = \{\emptyset, E, E^c, \mathbb{R}\}$ is the σ-algebra $\sigma(\{E\})$ of subsets of $\mathbb{R}$ generated by $\{E\}$.

   b) If $\mathcal{S}$ and $\mathcal{T}$ are collections of subsets of $\mathbb{R}$, is it true that

   $$\sigma(\mathcal{S} \cup \mathcal{T}) = \sigma(\mathcal{S}) \cup \sigma(\mathcal{T})?$$

   Prove or disprove. (5p)

2. On a measure space with measure μ we have measurable functions $\{f_n\}_{n\geq 1}$ and f. Show that the condition $\lim_{n\to\infty} \mu(|f_n - f| > 0) = 0$ implies that $\{f_n\}$ converges to f in μ-measure, but that the converse statement is not true. (5p)

3. Given an arbitrary sequence $\{x_n\}_1^\infty$ of real numbers, define

   $$g(x) = \sum_{n=1}^{\infty} e^{-n^2|x-x_n|}, \quad x \in \mathbb{R}$$

   Show that the function g is finite almost everywhere with respect to Lebesque measure. (6p)

4. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $f : X \to \mathbb{R}$ an integrable function in $L^1(X, \mu)$. Find the limit

   $$\lim_{n\to\infty} \frac{1}{n} \int_X \ln(1 + e^{-nf})\, d\mu.$$

   (6p)

5. Suppose g is a Lebesgue measurable real-valued function on $[0,1]$ such that the function $f(x, y) = 2g(x) - 3g(y)$ is Lebesgue integrable over $[0,1] \times [0,1]$. Show that g is Lebesgue integrable over $[0,1]$. (6p)

6. Let μ be a measure and f a measurable function on the real interval $[0,1]$, such that $0 < f(x) < \infty$ for all $x \in [0,1]$. Show that

   $$\int_{[0,1]} f^{-1}\, d\mu \geq \Big(\int_{[0,1]} f\, d\mu\Big)^{-1}. \quad (6p)$$

7. Let $(X, \mathcal{A}, \mu)$ be a measure space and fix $p \in [1, \infty)$. Let $\{f_n\}_{n\geq 1}$ and f be functions in $L^p(X, \mathcal{A}, \mu)$. Suppose $\lim_{n\to\infty} \|f_n - f\|_p = 0$. Show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $n \geq 1$ we have

   $$\int_E |f_n|^p\, d\mu < \varepsilon \quad \text{whenever } \mu(E) < \delta. \quad (6p)$$