UPPSALA UNIVERSITET
Matematiska Institutionen
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PROV I MATEMATIK
Integration Theory, 1MA215
2022-08-19

Grades 3, 4, 5 normally requires at least 18, 25, 32 credits.

1. Consider $\mathcal{F} = \{E \in \mathbb{R}$: either E is countable or E^c is countable.
   a) Show that $\mathcal{F}$ is a σ-algebra. (3)
   b) Find a measure μ on the measure space $(\mathbb{R}, \mathcal{F})$ such that the only μ-null set is $\emptyset$. (2)

2. Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers, and set $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$. Show that the function g is finite almost everywhere with respect to Lebesque measure on $\mathbb{R}$. (5)

3. Let μ be counting measure defined on the set of natural numbers $\mathbb{N} = \{1, 2, \dots\}$. Consider suitable sequences of real-valued measurable functions on this measure space and formulate and interpret the monotone and dominated convergence theorems and Fatou's lemma as statements about infinite series. (6)

4. Let $f_n(x) = ae^{-anx} - be^{-bnx}$, $x \geq 0$, where $0 < a < b$. Show that
   a) $\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\, dx = 0$;
   b) $\sum_{n=1}^{\infty} f_n(x)$ is integrable on $[0, \infty)$ with respect to Lebesgue measure;
   c) $\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dx = \log(b/a)$; and
   d) determine $\sum_{n=1}^{\infty} \int_0^{\infty} |f_n(x)|\, dx$. (6)

5. Let μ be a measure and f a measurable function on the real interval $[0, 1]$, such that $\mu([0, 1]) \geq 1$ and $0 < f(x) < \infty$ for all $x \in [0, 1]$. Prove the inequality
$$\int_{[0,1]} f^{-1}\, d\mu \geq \Big(\int_{[0,1]} f\, d\mu \Big)^{-1}. \quad (6p)$$

6. Let μ and ν be σ-finite measures on a measurable space $(X, \mathcal{A})$. Show that there exists measurable disjoint sets $A, B \in \mathcal{A}$, $A \cap B = \emptyset$, such that $X = A \cup B$ is a partition with μ and ν equivalent (mutually absolutely continuous) on A and singular measures on B, that is
$$\mu \sim \nu \quad \text{on} \quad (A, \mathcal{A} \cap A) \quad \text{and} \quad \mu \perp \nu \quad \text{on} \quad (B, \mathcal{A} \cap B). \quad (6p)$$
   Hint: consider $\mu + \nu$ as a reference measure

7. Let $(X, \mathcal{A}, \mu)$ be a measure space and fix $p \in [1, \infty)$. Let $\{f_n\}_{n \geq 1}$ and f be functions in $L^p(X, \mathcal{A}, \mu)$. Suppose $\lim_{n\to\infty} \|f_n - f\|_p = 0$. Show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $n \geq 1$ we have
$$\int_E |f_n|^p\, d\mu < \varepsilon \quad \text{whenever } \mu(E) < \delta. \quad (6p)$$