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Matematiska Institutionen
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PROV I MATEMATIK
Integration Theory, 1MA215
2023-01-10

Grades 3, 4, 5 normally requires at least 18, 25, 32 credits (including any bonus points).

1. Consider $\mathcal{F} = \{E \in \mathbb{R} :$ either E is countable or E^c is countable.
   a) Show that $\mathcal{F}$ is a σ-algebra. (3)
   b) Find a measure μ on $(\mathbb{R}, \mathcal{F})$ such that the only μ-null set is $\emptyset$. (2)

2. Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers, and set $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$. Show that g is finite almost everywhere with respect to Lebesque measure on $\mathbb{R}$. (5)

3. Let μ be a finite measure on a measurable space $(X, \mathcal{A})$. Let f and $f_1, f_2, \ldots$ be $\mathcal{A}$-measurable functions on X such that $f_n \to f$, μ-almost everywhere. Assume that the sequence $(f_n)_1^{\infty}$ also satisfies the property

$$\lim_{\alpha \to \infty} \sup_n \int_{\{|f_n| \geq \alpha\}} |f_n| \, d\mu = 0 \qquad (\star)$$

   a) Show that f is integrable. (2)
   b) Show that the sequence $h_n = |f_n - f|$ also satisfies property $(\star)$. (2)
   c) Show that $f_n \to f$ in $L^1(\mu)$. (2)

4. For $n \geq 1$, let

$$K_n = \frac{n}{1 - e^{-n}} \int_0^1 \ln\Big(\frac{1}{y} - 1\Big) e^{-ny} \, dy.$$

   Show that there is a constant C, such that

$$K_n - \ln n \to \mathrm{C}, \quad n \to \infty. \qquad (6)$$

   [Can you identify C and give an approximate numerical value? 1 *bonus point*]

5. Let μ and ν be σ-finite measures on a measurable space $(X, \mathcal{A})$. Show that there exists measurable disjoint sets $A, B \in \mathcal{A}$, $A \cap B = \emptyset$, such that $X = A \cup B$ is a partition with μ and ν equivalent (mutually absolutely continuous) on A and singular measures on B, that is

$$\mu \sim \nu \quad \text{on} \quad (A, \mathcal{A} \cap A) \quad \text{and} \quad \mu \perp \nu \quad \text{on} \quad (B, \mathcal{A} \cap B). \qquad (6)$$

   Hint: consider $\mu + \nu$ as a reference measure

V.g.v.

6. Let $f_n(x) = ae^{-anx} - be^{-bnx}$, $x \geq 0$, where $0 < a < b$. Show that

a) $\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\, dx = 0$;

b) $\sum_{n=1}^{\infty} f_n(x)$ is integrable on $[0, \infty)$ with respect to Lebesgue measure;

c) $\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dx = \log(b/a)$; and

d) determine $\sum_{n=1}^{\infty} \int_0^{\infty} |f_n(x)|\, dx$. (6)

7. Let $(X, \mathcal{A}, \mu)$ be a measure space and fix $p \in [1, \infty)$. Let $\{f_n\}_{n \geq 1}$ and f be functions in $L^p(X, \mathcal{A}, \mu)$. Suppose $\lim_{n \to \infty} \|f_n - f\|_p = 0$. Show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $n \geq 1$ we have

$$\int_E |f_n|^p \, d\mu < \varepsilon \quad \text{whenever } \mu(E) < \delta. \qquad (6)$$