UPPSALA UNIVERSITET
Matematiska Institutionen
Ingemar Kaj

PROV I MATEMATIK
Integration Theory, 1MA215
2023-06-08

Grades 3, 4, 5 normally requires at least 18, 25, 32 credits.

1. Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be a function in $L^1(X, \mathcal{A}, \mu)$. Show that if a is a real number and
$$\int_E f\,d\mu \le a\,\mu(E), \quad \text{all } E \in \mathcal{A},$$
then $f \le a$, μ-almost everywhere. (5p)

2. Given an arbitrary sequence $\{x_n\}_1^\infty$ of real numbers, define
$$g(x) = \sum_{n=1}^\infty e^{-n^2|x-x_n|}, \quad x \in \mathbb{R}$$
Show that the function g is finite almost everywhere with respect to Lebesque measure. (5p)

3. Let μ be counting measure defined on the set of natural numbers $\mathbb{N} = \{1, 2, \dots\}$. Consider suitable sequences of real-valued measurable functions on this measure space and formulate and interpret the monotone and dominated convergence theorems and Fatou's lemma as statements about infinite series. (6p)

4. Let f be an integrable function on $[0, a]$ and define $I^\alpha f$ by
$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(y)(x-y)^{\alpha-1}\,dy, \quad 0 \le x \le a.$$
Show that $I^\alpha f$ is well-defined and integrable for $\alpha > 0$. If $f \in L^2([0, a], m)$ show that $I^\alpha f \in L^2([0, a], m)$. (6p)

5. Suppose g is a Lebesgue measurable real-valued function on $[0, 1]$ such that the function $f(x, y) = 2g(x) - 3g(y)$ is Lebesgue integrable over $[0, 1] \times [0, 1]$. Show that g is Lebesgue integrable over $[0, 1]$. (6p)

6. Let $(X, \mathcal{A})$ be a measurable space and suppose μ and ν are probability measures, $\mu(X) = \nu(X) = 1$, such that the signed measure $\mu - \nu$ has total variation norm $|\mu - \nu|(X) = 2$.

   a) Show that $\mu \perp \nu$. (3p)

   b) Show that $|\mu - \nu| \ll \mu + \nu$. (3p)

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function and define
$$g(x) = \int_{-\infty}^\infty e^{-|y|} f(x-y)\,dy, \quad x \in \mathbb{R}.$$

   a) Suppose $(c_n)_{n\ge1}$ is a sequence of real numbers which converges to a real number c. Show the sequential continuity $g(c_n) \to g(c)$, $n \to \infty$. (3p)

   b) Show that g is a function of bounded variation. (3p)