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Matematiska Institutionen
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PROV I MATEMATIK
Integration Theory, 1MA215
2023-08-22

Grades 3, 4, 5 normally requires at least 18, 25, 32 credits.

1. Find
$$\lim_{n\to\infty}\int_0^{n^2}\frac{2x}{n^2}\left(1-\frac{x}{n^2}\right)^n dx. \qquad \text{(5p)}$$

2. Given an arbitrary sequence $\{x_n\}_1^\infty$ of real numbers, define
$$g(x)=\sum_{n=1}^{\infty}\frac{1}{n^2\sqrt{|x-x_n|}}, \qquad x\in\mathbb{R}.$$
Show that g is finite almost everywhere with respect to Lebesque measure. (5p)

3. Let $(X,\mathcal{A},\mu)$ be a measure space and let f and g be functions in $L^1(X,\mathcal{A},\mu)$. Assume that there is a sequence $(f_n)_{n\geq 1}$ of functions in $L^1(X,\mathcal{A},\mu)$, such that f_n converges to f μ-almost everywhere and f_n converges to g in L^1-norm. Prove that $f=g$ almost everywhere with respect to μ. (6p)

4. Suppose that f is an absolutely continuous function on $[0,1]$ such that $f(0)=0$ and $f'\in L^4([0,1],m]$. Prove that
$$\int_0^1|f(x)|^4\,dx\leq\frac{1}{4}\int_0^1|f'(x)|^4\,dx \qquad \text{(6p)}$$

5. The function F is given by
$$F(x)=\begin{cases}0, & x<-1\\ 1, & -1\leq x<2\\ x, & x\geq 2.\end{cases}$$
Let $\mu=\mu_F$ be the associated measure defined by $\mu_F((-\infty,x])=F(x)$, $x\in\mathbb{R}$. Determine the Lebesgue decomposition of μ and find $\int_{[-3,3]}x\,d\mu(x)$. (6)

6. Suppose that μ and ν are two finite measures on the same measurable space $(X,\mathcal{F})$.
   a) Show that if ρ is a σ-finite measure on $(X,\mathcal{F})$ such that $\mu\ll\rho$ and $\nu\ll\rho$, then
   $$\int_X\sqrt{\frac{d\mu}{d\rho}\frac{d\nu}{d\rho}}\,d\rho<\infty. \qquad \text{(2p)}$$
   b) Show that there exists at least on such dominating measure ρ. (1p)
   c) Show that the integral in a) is independent of the choice of ρ. (3p)

7. Assume that $f,g\in L^2(\mathbb{R}^d,m)$, that is, f and g are square-integrable functions with respect to Lebesgue measure m defined on $\mathbb{R}^d$. Put $F(x,y)=(f(x)g(y)-f(y)g(x))^2$, $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$.
   a) Show that the function F is integrable on the product space $\mathbb{R}^d\times\mathbb{R}^d$ with respect to the product measure $m\otimes m$. (3p)
   b) Use the result in a) to prove the Cauchy-Schwarz intequality in $\mathbb{R}^d$, namely
   $\|fg\|_{L^1(\mathbb{R}^d)}\leq\|f\|_{L^2(\mathbb{R}^d)}\,\|g\|_{L^2(\mathbb{R}^d)}$. (3p)