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Prov i matematik
Integrationsteori 1MA215
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Solutions may be written in English or Swedish.
Points are given after each problem. You need 18 points for grade 3, 27 points for grade 4, and 36 points for grade 5 (including any bonus points).

1. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $E \subseteq X$ be a measurable subset. Define $\mathcal{A}_E := \{A \in \mathcal{A} : A \subseteq E\}$. Show that:

(a) $\mathcal{A}_E$ is a σ-algebra on E. (2p)

(b) If μ_E is the restriction of μ to $\mathcal{A}_E$, then μ_E is a measure on E. (2p)

(c) If $f : X \to \mathbb{C}$ is an $\mathcal{A}$-measurable function on X, then its restriction to E is $\mathcal{A}_E$-measurable. (2p)

(d) If $f : X \to \mathbb{C}$ is an $\mathcal{A}$-measurable function on X, then the restriction of f to E is μ_E-integrable if and only if $f\chi_E$ is μ-integrable on X, and then

$$\int_E f \, \mathrm{d}\mu_E = \int_X f\chi_E \, \mathrm{d}\mu. \qquad (2p)$$

2. Let f be a non-negative measurable function $\mathbb{R} \to [0, \infty)$. Suppose that for every integer n,

$$\int_{\mathbb{R}} \frac{n^2}{x^2 + n^2} f(x) \, \mathrm{d}x \le 1.$$

Show that f is integrable and $\|f\|_1 \le 1$. (5p)

3. Let μ be a finite measure on $\mathbb{R}$, and define its Fourier transform by

$$\widehat{\mu}(t) := \int_{\mathbb{R}} e^{-\mathrm{i}tx} \, \mathrm{d}\mu(x), \qquad t \in \mathbb{R}.$$

Show that the function $\widehat{\mu}(t)$ is bounded and continuous. (5p)

4. Let f be a function on $[0, 1]$ and let $M \ge 0$. Show that f satisfies the condition

$$|f(x) - f(y)| \le M|x - y|$$

for all $x, y \in [0, 1]$ if and only if there is a bounded measurable function g with $|g(x)| \le M$ for all x such that

$$f(x) = f(0) + \int_0^x g(y) \, \mathrm{d}y. \qquad (6p)$$

p.t.o.

5. Let $f_n(x) := (\sin 2\pi n x)^{-1/3}$ (interpreted as $+\infty$ when $\sin 2\pi n x = 0$). Show that if $\alpha > 1$, then

$$\sum_{n=1}^{\infty} n^{-\alpha} f_n(x)$$

converges for a.e. real x. (5p)

6. Find the limit

$$\lim_{t \searrow 0} \int_0^\infty e^{-x} \frac{t}{t^2 + x^2} \, dx. \qquad (5p)$$

7. Let μ be a σ-finite measure on $\mathbb{R}$. Let A be a subset of $\mathbb{R}$ with Lebesgue measure $m(A) = 0$, and let $A + x := \{a + x : a \in A\}$ denote the translate of A by x. Show that $\mu(A + x) = 0$ for m-a.e. x.
Give an example of μ and A such that $\mu(A + x) > 0$ holds for some x. (6p)

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