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Prov i matematik
Integrationsteori 1MA215
2024-08-26

Solutions may be written in English or Swedish.
Points are given after each problem. You need 18 points for grade 3, 25 points for grade 4, and 32 points for grade 5.

1. Let E be the set of all numbers in $(0,1)$ that have a decimal expansion with no 9. Show that E is measurable and that its Lebesgue measure $m(E) = 0$. (5p)

2. Let A and B be Lebesgue measurable subsets of $(0,1)$ such that $m(A) > 1/2$ and $m(B) > 1/2$. Prove that there exist $x \in A$ and $y \in B$ such that $x + y = 1$. (6p)

3. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f_1, f_2, \dots$ be a sequence of complex-valued integrable functions on X such that the set of integrals $\int_X |f_n| \, \mathrm{d}\mu$ is bounded. (In other words, there exists a constant C with $\int_X |f_n| \, \mathrm{d}\mu \le C$ for every $n \ge 1$.) Suppose that f is a function such that $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$. Show that f is integrable. (5p)

4. Let f be an integrable function on a measure space $(X, \mathcal{F}, \mu)$.

(a) Show that

$$t\mu\bigl\{x \in X : |f(x)| > t\bigr\} \to 0 \qquad \text{as } t \to 0. \tag{3p}$$

(b) Show that

$$t\mu\bigl\{x \in X : |f(x)| > t\bigr\} \to 0 \qquad \text{as } t \to \infty. \tag{3p}$$

5. Compute

$$\lim_{n\to\infty} n \int_0^1 (1-x)^n e^{nx/2} \, \mathrm{d}x. \tag{6p}$$

p.t.o.

6. Let μ and ν be finite measures on some measurable space $(X, \mathcal{A})$, and let $\lambda = \mu + \nu$.

(a) Show that there exists a measurable function $f : X \to [0, 1]$ such that, for every measurable $A \subseteq X$,

$$\mu(A) = \int_A f \, \mathrm{d}\lambda, \qquad \nu(A) = \int_A (1 - f) \, \mathrm{d}\lambda. \qquad (2p)$$

(b) Show that μ is absolutely continuous with respect to ν ($\mu \ll \nu$) if and only if $f < 1$ λ-a.e. on X. (2p)

(c) Show that μ and ν are mutually singular ($\mu \perp \nu$) if and only if $f(x) \in \{0, 1\}$ for λ-a.e. $x \in X$. (2p)

7. Suppose that $A \subseteq [0, 1]$ is a measurable set such that $m(I \cap A) \leq \frac{1}{2} m(I)$ for all intervals $I \subseteq [0, 1]$. Prove that $m(A) = 0$. (6p)

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