UPPSALA UNIVERSITET
Matematiska Institutionen
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PROV I MATEMATIK
Integration Theory, 1MA215
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Good reasoning and justifications will be taken into account during assessment and scoring. Grades 3, 4, 5 normally requires at least 18, 25, 32 credits (including any bonus points)

1. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let f be a function in $L^1(X, \mathcal{A}, \mu)$. Show that if a is a real number and
$$\int_E f\, d\mu \leq a\, \mu(E), \quad \text{for all } E \in \mathcal{A},$$
then $f \leq a$, μ-almost everywhere. (5p)

2. Compute
$$\lim_{n\to\infty} n \int_0^1 (1+x)^{-n}(1-\sin x)\, dx. \qquad (6p)$$

3. Consider the measure space $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu)$, where $\mathcal{P}(\mathbb{Z})$ is the collection of all subsets of the integers $\mathbb{Z}$, and μ is the counting measure, $\mu(A) =$ the number of integers in A, $A \in \mathcal{P}(\mathbb{Z})$. Let f and $f_1, f_2, \ldots$ be real-valued functions on $\mathbb{Z}$. Prove that f_n converges in μ-measure to f if and only if f_n converges uniformly to f. (6p)

4. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $f, g \in L^2(X, \mathcal{A}, \mu)$. Define a function $F : X \times X \to \mathbb{R}$ by
$$F(x, y) = (f(x)g(y) - f(y)g(x))^2, \quad (x, y) \in X \times X.$$
Show that F is integrable on the product space $(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \otimes \mu)$ and use this result to prove the Cauchy-Schwarz inequality for the Hilbert space $L^2(X, \mathcal{A}, \mu)$. (6p)

5. Let μ and ν be σ-finite measures on a measurable space $(X, \mathcal{A})$. Show that there exists measurable disjoint sets $A, B \in \mathcal{A}$, $A \cap B = \emptyset$, such that $X = A \cup B$ is a partition with μ and ν equivalent (mutually absolutely continuous) on A and singular measures on B, that is
$$\mu \sim \nu \quad \text{on} \quad (A, \mathcal{A} \cap A) \quad \text{and} \quad \mu \perp \nu \quad \text{on} \quad (B, \mathcal{A} \cap B). \qquad (6)$$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function and define
$$G(x) = \int_{-\infty}^{\infty} e^{-|y|} f(x-y)\, dy, \quad x \in \mathbb{R}.$$
Show that there exists a unique signed measure μ_G on $\mathbb{R}$, such that $\mu_G((-\infty, x]) = G(x)$, $x \in \mathbb{R}$. (6p)

7. Suppose that f is an absolutely continuous function on $[0, 1]$ such that $f(0) = 0$ and $f' \in L^4([0, 1], m]$. Prove that
$$\frac{|f(t)|^4}{t^3} \to 0, \quad \text{as} \quad t \to 0$$
and, for every $\varepsilon > 0$,
$$\int_0^1 \frac{|f(t)|^4}{t^{4-\varepsilon}}\, dt \leq \frac{1}{\varepsilon} \int_0^1 |f'(y)|^4\, dy \qquad (5p)$$