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PROV I MATEMATIK
Integration Theory, 1MA215
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Solutions

1. Set $g = f - a$ so that

$$\int_E g\,d\mu \le 0, \quad \text{all } E \in \mathcal{A}.$$

Now choose $E = \{g > 0\}$ to obtain

$$0 \ge \int_E g\,d\mu = \int_{\{g>0\}} g\,d\mu = \int_X g\,1_{\{g>0\}}\,d\mu \ge 0$$

as $g\,1_{\{g>0\}} \ge 0$, μ-almost everywhere. Hence

$$\int_X g\,1_{\{g>0\}}\,d\mu = 0.$$

It follows that

$$E = \{g\,1_{\{g>0\}} > 0\} = \bigcup_{n=1}^{\infty} \Big\{g\,1_{\{g>0\}} > \frac{1}{n}\Big\}$$

is a μ-null set, since

$$\mu(\{g\,1_{\{g>0\}} > 1/n\}) \le n \int_X g\,1_{\{g>0\}}\,d\mu = 0$$

and a countable union of null sets is a null set (subadditivity of measures). Hence $f \le a$, μ-a.e.

Alternative: Assume $E = \{g > 0\}$ has positive measure, $\mu(E) > 0$. Then

$$\int_E f\,d\mu > \int_E a\,d\mu = a\mu(E) \ge \int_E f\,d\mu,$$

again using the special assumption in this problem. This is a contradiction and the conclusion is $\mu(E) = 0$.

2. To find the limit

$$\lim_{n\to\infty} n \int_0^1 (1+x)^{-n}(1-\sin x)\,dx,$$

a variable substitution from x to nx yields

$$n \int_0^1 (1+x)^{-n}(1-\sin x)\,dx = \int_0^\infty f_n(x)\,dx, \quad f_n(x) = 1_{\{x\le n\}}\Big(1+\frac{x}{n}\Big)^{-n}\Big(1-\sin\frac{x}{n}\Big).$$

For every fixed $x > 0$, as $n \to \infty$, $1_{\{x\le n\}} \to 1$, $(1+x/n)^{-n} \to e^{-x}$ and $1-\sin(x/n) \to 1$, and thus $f_n(x) \to e^{-x}$.

We want to apply dominated convergence, and we thus seek a dominating function that is integrable.

Method 1. Use the Taylor series expansion, for $0 \le x \le 1$,

$$\ln(1+x) = \Big(x - \frac{x^2}{2}\Big) + \Big(\frac{x^3}{3} - \frac{x^4}{4}\Big) + \cdots \ge x - \frac{x^2}{2} \ge x/2.$$

Hence, for $0 \le x \le n$,

$$\Big(1+\frac{x}{n}\Big)^{-n} = e^{-n\ln(1+x/n)} \le e^{-nx/2n} = e^{-x/2}$$

and thus, since $f_n(x) = 0$ for $x > n$,

$$|f_n(x)| \leq 2\, e^{-x/2}, \quad x \in \mathbb{R},\ n \geq 1.$$

Mathod 2. Use that $(1 + x/n)^n$ is an increasing function in n, for every $x \geq 0$. Thus,

$$\Big(1 + \frac{x}{n}\Big)^{-n} \leq \Big(1 + \frac{x}{2}\Big)^{-2} \leq \frac{1}{1+x},$$

and

$$|f_n(x)| \leq 2(1 + x/2)^{-2}, \quad x \geq 0,$$

which is integrable (the rightmost bound above is *not* integrable).

Method 3. For example, use the binomial theorem to find a lower bound of $(1 + x/n)^n$.

Now, the dominated convergence theorem applies and we conclude that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx = \int_0^\infty \lim_{n\to\infty} f_n(x)\, dx = \int_0^\infty e^{-x}\, dx = 1.$$

Alternative. Rewrite as

$$n \int_0^1 (1+x)^{-n}(1 - \sin x)\, dx = n \int_0^1 (1+x)^{-n}\, dx - n \int_0^1 (1+x)^{-n} \sin x\, dx,$$

evluate the first integral and apply the DCT to the second using $\sin x \leq x$ to obtain a dominating function, which is integrable.

3. Since every subset is measurable, all functions f and f_n are measurable with respect to $\mathcal{F}$.

   A sequence f_n is said to converge to f in measure with respect to μ, if $\mu(\{|f_n - f| > \varepsilon\}) \to 0$ as $n \to \infty$, for each $\varepsilon > 0$. In particular, if f_n converges to f in counting measure μ we can find N such that $\mu(\{z \in \mathbb{Z} : |f_n(z) - f(z)| > \varepsilon\}) < 1/2$ whenever $n \geq N$. But then $|f_n(z) - f(z)| \leq \varepsilon$ for each integer z, and hence $\sup_{z\in\mathbb{Z}} |f_n(z) - f(z)| \leq \varepsilon$, for $n \geq N$.

   Conversely, assuming f_n converges uniformly to f, fix ε and take n so large that $\sup_{z\in\mathbb{Z}} |f_n(z) - f(z)| \leq \varepsilon$. Then $\{z : |f_n(z) - f(z)| > \varepsilon\}$ is a μ-zero set for such n and hence, obviously, $\mu\{z : |f_n(z) - f(z)| > \varepsilon\} \to 0$ as $n \to \infty$.

4. Writing

$$F(x,y) = f^2(x)g^2(y) + f^2(y)g^2(x) - G(x,y), \quad G(x,y) = 2f(x)f(y)g(x)g(y),$$

we observe that

$$|G(x,y)| \leq f^2(x)g^2(y) + f^2(y)^2g^2(y)$$

and

$$F(x,y) \leq f^2(x)g^2(y) + f^2(y)^2g^2(y) + |G(x,y)| \leq 2(f^2(x)g^2(y) + f^2(y)^2g^2(y)).$$

Hence

$$\int_{X\times X} |G(x,y)|\, d(\mu\otimes\mu) \leq 2 \int_{X\times X} f^2g^2\, d(\mu\otimes\mu),$$

where the factor 2 is from the symmetry in the variables x and y, and

$$\int_{X\times X} F(x,y)\, d(\mu\otimes\mu) \leq 4 \int_{X\times X} f^2g^2\, d(\mu\otimes\mu).$$

As the measure space is σ-finite we can apply Fubini's theorem. Fubini's theorem for non-negative functions (Fubini-Tonelli) implies

$$\int_{X\times X} f^2g^2\,d(\mu\otimes\mu) = \int_X f^2(x)\Big(\int_X g^2(y)\,d\mu(y)\Big)\,d\mu(x) = \int_X f^2\,d\mu\int_X g^2\,d\mu < \infty,$$

since f and g are square-integrable, Thus, we have shown that F is integrable on the product space. Moreover, $|G|$ is integrable and hence G is. By the regular Fubini's theorem for signed functions, we obtain

$$\int_{X\times X} G(x,y)\,d(\mu\otimes\mu) = 2\int_X fg\,d\mu\int_X fg\,d\mu < \infty.$$

Furthermore,

$$\int_{X\times X} F(x,y)\,d(\mu\otimes\mu) = 2\Big(\int_X f^2\,d\mu\int_X g^2\,d\mu - \int_X fg\,d\mu\int_X fg\,d\mu\Big) \geq 0$$

and therefore

$$\Big|\int_X fg\,d\mu\Big| \leq \sqrt{\int_X f^2\,d\mu}\sqrt{\int_X g^2\,d\mu},$$

which is the desired Cauchy-Schwarz inequality.

5. Let λ be the σ-finite measure $\lambda = \mu + \nu$. Then $\mu \ll \lambda$ and $\nu \ll \lambda$. By the Radon-Nikodym theorem there exists nonnegative functions g and h such that for every E in $\mathcal{A}$,

$$\mu(E) = \int_E g\,d\lambda \quad \text{and} \quad \nu(E) = \int_E h\,d\lambda.$$

Let $A = \{x \in X : g(x)h(x) > 0\}$ and $B = A^c$. If $E \in \mathcal{A}$ and $E \subset A$ then $\mu(E)$ implies $\lambda(E) = 0$ since $g > 0$ on A, and therefore, $\nu(E) = 0$. Thus $\nu \ll \mu$ on A. By symmetry we can prove $\mu \ll \nu$ on A in the same manner. Hence, μ and ν are equivalent on A.

Next, we partition B as $B = C \cup D$, where

$$C = \{x : h(x) = 0\}, \quad D = B \setminus C = \{x : h(x) > 0, g(x) = 0\}.$$

For all measurable sets $E \subset C$ and $F \subset D$ we then have $\mu(E) = \nu(F) = 0$. Hence, indeed, μ and ν are singular measures on B.

6. We have

$$G(x) = \int_{-\infty}^{\infty} e^{-|y|}f(x-y)\,dy, \quad x \in \mathbb{R},$$

where $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue integrable function. Then there exists a unique signed measure μ_G on $\mathbb{R}$, such that $\mu_G((-\infty, x]) = G(x)$, $x \in \mathbb{R}$, if and only if 1) G is right-continuous, 2) G satisfies $G(x) \to 0$ as $x \to -\infty$, and 3) G has bounded variation.

Letting h be the exponential function $h(x) = e^{-|x|}$, G is the convolution $G = h * f$. By a general result, the convolution of a continuous function with an integrable function is continuous, which implies the weaker property 1). This follows from Lebesgue's dominated convergence theorem as sequential continuity. Indeed, let $(c_n)_{n\geq 1}$ be a sequence of real numbers that converges to a real number c. Then

$$G(c_n) = \int_{\mathbb{R}} e^{-|c_n-y|}f(y)\,dy \to \int_{\mathbb{R}} e^{-|c-y|}f(y)\,dy = G(c).$$

Similarly, 2) follows from the DCT, as $G(c_n) \to 0$ whenever $c_n \to -\infty$.

To prove 3), let $\{x_k\}$ be a partition of the real line, $-\infty < x_0 < x_1 < \cdots < x_n < \infty$. Then

$$\sum_{k=1}^{n} |G(x_k) - G(x_{k-1})| \leq \int_{\mathbb{R}} \sum_{k=1}^{n} |h(x_k - y) - h(x_{k-1} - y)|\, |f(y)|\, dy.$$

Now, the exponential function h has bounded variation. One way to see this is to write h as a difference of two increasing functions. Thus, taking the supremum over all partitions, there is a finite constant C such that

$$\sup_{\{x_k\}} |h(x_k) - h(x_{k-1})| \leq C.$$

Hence

$$\sup_{\{x_k\}} \sum_{i=1}^{n} |G(x_k) - G(x_{k-1})| \leq C \int_{\mathbb{R}} |f(y)|\, dy < \infty,$$

which completes the proof of 3) and hence the proof of existence of μ_G.

7. Since f is an absolutely continuous function on $[0,1]$ with $f(0) = 0$,

$$f(t) = \int_0^t f'(y)\, dy,$$

where f' exists almost everywhere. By Hölder's inequality with conjugate exponents $p = 4$ and $q = 4/3$,

$$|f(t)| \leq \int_0^t |f'(y)|\, dy \leq \Big(\int_0^t |f'(y)|^4\, dy \Big)^{1/4} t^{3/4} < \infty,$$

since $f' \in L^4$. Moreover,

$$\frac{|f(t)|^4}{t^3} \leq \int_0^t |f'(y)|^4\, dy \to 0, \quad \text{as} \quad t \to 0.$$

Also, for every $\varepsilon > 0$,

$$\frac{|f(t)|^4}{t^{4-\varepsilon}} \leq \int_0^t |f'(y)|^4\, dy\, t^{\varepsilon - 1}, \quad t > 0,$$

and therefore, using Fubini's theorem,

$$\begin{aligned} \int_0^1 \frac{|f(t)|^4}{t^{4-\varepsilon}}\, dt &\leq \int_0^1 \int_0^t |f'(y)|^4\, dy\, t^{\varepsilon-1}\, dt \\ &= \int_0^1 |f'(y)|^4 \int_y^1 t^{\varepsilon-1}\, dt\, dy \leq \frac{1}{\varepsilon} \int_0^1 |f'(y)|^4\, dy < \infty. \end{aligned}$$