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**Matematiska institutionen**
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Prov i matematik
KandMa2, Frist
KOMPLEX ANALYS 10hp
2014–08–26

*Writing time: 14.00 – 19.00. Tools allowed: pens, pencils, rubber. Every correctly solved problem gives up to* 5 *points.*

**1.** Determine all analytic functions $f = u + iv$ with the real part of the form

$$u(x, y) = e^y \varphi(x) \ ,$$

where $\varphi$ is a $C^2$-function of one real variable. The answer should be expressed as a function of the variable $z = x + iy$.

**2.** Find a conformal and bijective mapping of the region $U = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0 \text{ and } |z| > 1\}$ onto the upper half-plane $\mathrm{Im}(w) > 0$.

**3.** Find the Laurent series of the function

$$f(z) = \frac{1}{z^2(z-1)}$$

in the region $1 < |z - i| < \sqrt{2}$ around the point $z_0 = i$.

**4.** Calculate the value of the integral

$$I = \int_0^\infty \frac{\cos x}{(x^2+1)^2} \, dx \ .$$

**5.** Determine the number of zeros of the polynomial $P(z) = z^{10} + iz^6 + 1$ in the first quadrant $\mathrm{Re} z > 0$, $\mathrm{Im} z > 0$.

**6.** Find all functions $f(z)$ which are analytic in $\mathbb{C}$ except for a pole of order 2 at $z_0 = 1$, with the residue 1 at $z_0 = 1$, which have a zero of order $\geq 1$ at $z_1 = -1$ and which satisfy $\lim_{z\to\infty} f(z) = 1$. (The function $f(z)$ has to satisfy all these conditions at the same time).

**(Continued on the next page)**

**7.** Suppose that functions $f$ and $g$ are analytic on the disc $A = \{z : |z| < 2\}$ and that neither $f(z)$ nor $g(z)$ is ever $0$ for $z \in A$. If

$$\frac{f'(\frac{1}{n})}{f(\frac{1}{n})} = \frac{g'(\frac{1}{n})}{g(\frac{1}{n})} \qquad \text{for} \quad n = 1, 2, \dots ,$$

show that there is a constant $c$ such that $f(z) = c\, g(z)$ for all $z \in A$.

**8.** Let $D$ be a domain in $\mathbb{C}$ bounded by a simple closed curve and let $z_0 \in D$. If $f(z)$ is analytic on $D \cup \partial D$ except for a simple pole at $z_0$ and $|f(z)| = 1$ on $\partial D$, show that $f$ takes in $D$ exactly once every value $a$ with $|a| > 1$. (Hint: consider functions $g(z) = -a$ and $h(z) = f(z) - a$.)

**Good Luck!**

**Svar till tentamen i KOMPLEX ANALYS 10hp 2014–08–26**

**1.** $f(z) = De^{-iz} + ic, \quad D \in \mathbb{C},\ c \in \mathbb{R}\,.$

**2.** F.ex. $F(z) = \left(\dfrac{z-1}{z+1}\right)^2.$

**3.** $f(z) = -\displaystyle\sum_{m=1}^{\infty} \frac{(-i)^{m-2}(m-1-i)}{(z-i)^m} - \sum_{n=1}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}\,.$

**4.** $I = \dfrac{\pi}{2e}\,.$

**5.** 3 zeros.

**6.** $f(z) = \dfrac{(z+1)(z-2)}{(z-1)^2}\,.$