UPPSALA UNIVERSITET
Matematiska institutionen
M. Klimek

Prov i matematik
Kurs: 1MA022
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Complex Analysis

Writing time: 14:00–19:00.
Other than writing utensils and paper, no help materials are allowed.

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1. Suppose that $u(x, y)$ and $v(x, y)$ are harmonic functions in a domain $D \subset \mathbb{C}$. Let $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy \in D$. Show that if the function f is analytic in D, then the product $u(x, y)v(x, y)$ is harmonic in D. Is the converse statement true?

2. Find a Möbius transformation that maps the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ onto the circle $\{z \in \mathbb{C} : |z - 1| = 1\}$, while mapping the points 0 and 1 onto the points 5/2 and 0, respectively.

3. Find the Laurent series expansion of the function

$$f(z) = \frac{1}{(z-i)} + \frac{1}{(z+2i)^2}$$

in the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

4. Let

$$S = \{z \in \mathbb{C} \, : \, |\mathrm{Re}\, z| < \pi \text{ and } |\mathrm{Im}\, z| < \pi\}$$

and let

$$D = \{z \in \mathbb{C} : e^{-\pi} < |z| < e^{\pi} \text{ and } |\mathrm{Arg}\, z| < \pi\}.$$

Show that for any given $w \in D$, the function

$$f(z) = \frac{ze^z}{e^z - w}$$

has only one simple pole within the square S. Prove that

$$\mathrm{Log}\, w = \frac{1}{2\pi i} \int_{\partial S} f(z)dz.$$

5. Use the residue theorem to show that

$$\int_0^\infty \frac{2\sin^2 x}{x^2}\,dx = \pi.$$

Hint: Note that $2\sin^2 x = \mathrm{Re}\,(1 - e^{2ix})$ for $x \in \mathbb{R}$.

6. Show that the zeros of the polynomial $p(z) = z^4 - 2iz^3 + 16$ are contained in the disc $\{z \in \mathbb{C} : |z| < 3\}$. For how many zeros both the real and imaginary parts are negative?

7. Suppose that $f : \mathbb{C} \setminus \{1\} \longrightarrow \mathbb{C}$ is an analytic function which has the following properties:

- f has pole of order 3 at 1, with residue 2;
- f has double zeros at $\pm i$;
- f has a simple pole at ∞.

Find an explicit formula for such a function. Can there be more than one function with these properties?

8. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic function such that for some constant $A > 0$ the inequality

$$|f(z)| \leq A + \sqrt{|z|}$$

is satisfied for all $z \in \mathbb{C}$. Show that f has to be a constant function.

GOOD LUCK!