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Matematiska institutionen
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Prov i matematik
Kurs: 1MA022
2015-08-27

Complex Analysis

Writing time: 14:00–19:00.
Other than writing utensils and paper, no help materials are allowed.

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1. Suppose that $u(x,y)$ and $v(x,y)$ are harmonic functions in a domain $D \subset \mathbb{C}$. Let $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy \in D$. Show that if the function f is analytic in D, then the product $u(x,y)v(x,y)$ is harmonic in D. Is the converse statement true?

2. Find a Möbius transformation that maps the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ onto the circle $\{z \in \mathbb{C} : |z-1| = 1\}$, while mapping the points 0 and 1 onto the points 5/2 and 0, respectively.

3. Find the Laurent series expansion of the function

$$f(z) = \frac{1}{(z-i)} + \frac{1}{(z+2i)^2}$$

in the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

4. Let

$$S = \{z \in \mathbb{C} \,:\, |\operatorname{Re} z| < \pi \text{ and } |\operatorname{Im} z| < \pi\}$$

and let

$$D = \{z \in \mathbb{C} : e^{-\pi} < |z| < e^{\pi} \text{ and } |\operatorname{Arg} z| < \pi\}.$$

Show that for any given $w \in D$, the function

$$f(z) = \frac{ze^z}{e^z - w}$$

has only one simple pole within the square S. Prove that

$$\operatorname{Log} w = \frac{1}{2\pi i}\int_{\partial S} f(z)dz.$$

5. Use the residue theorem to show that

$$\int_0^\infty \frac{2\sin^2 x}{x^2}\,dx = \pi.$$

Hint: Note that $2\sin^2 x = \mathrm{Re}\,(1 - e^{2ix})$ for $x \in \mathbb{R}$.

6. Show that the zeros of the polynomial $p(z) = z^4 - 2iz^3 + 16$ are contained in the disc $\{z \in \mathbb{C} : |z| < 3\}$. For how many zeros both the real and imaginary parts are negative?

7. Suppose that $f : \mathbb{C} \setminus \{1\} \longrightarrow \mathbb{C}$ is an analytic function which has the following properties:

- f has pole of order 3 at 1, with residue 2;
- f has double zeros at $\pm i$;
- f has a simple pole at ∞.

Find an explicit formula for such a function. Can there be more than one function with these properties?

8. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic function such that for some constant $A > 0$ the inequality

$$|f(z)| \leq A + \sqrt{|z|}$$

is satisfied for all $z \in \mathbb{C}$. Show that f has to be a constant function.

GOOD LUCK!

SOLUTIONS

Solution 1: It is easy to check that in general $\Delta(uv) = u\Delta v + v\Delta u + 2(u_x v_x + u_y v_y)$. Since our u and v are harmonic $\Delta(uv) = 2(u_x v_x + u_y v_y)$. If $f = u + iv$ is analytic, then the Cauchy-Riemann equations hold: $u_x = v_y, u_y = -v_x$, and hence $\Delta(uv) = 0$. A counterexample for the converse statement is given by $u(x, y) = x$ and $v(x, y) = -y$, as $f(z) = \bar{z}$ is not analytic.

Solution 2: Since $D(1, 1)$ is the unit disc shifted to the right by 1 unit, we basically want a Möbius transformation that maps the unit circle onto itself and the points 0,1 onto the points 3/2, -1, respectively. The inversion $z \mapsto 1/z$ swaps the interior with the exterior of the unit circle and 3/2 with 2/3, while keeping -1 fixed. So we also need a Möbius transformation mapping the unit disc onto itself and the points 0,1 onto the points 2/3, -1, respectively. We know that this is given by the formula

$$T_{2/3}(z) = \frac{z - 2/3}{2z/3 - 1}.$$

So the required mapping is

$$f(z) = \frac{1}{T_{2/3}(z)} + 1.$$

Solution 3: If $|z| > 1$, then

$$\frac{1}{z - i} = \frac{1}{z}\frac{1}{1 - \frac{i}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{i^n}{z^n} = \sum_{n=1}^{\infty}\frac{i^{n-1}}{z^n}.$$

Observe that

$$\frac{1}{(z + 2i)^2} = \left(\frac{-1}{z + 2i}\right)'.$$

If $|z| < 2$, then

$$\frac{-1}{z + 2i} = \frac{-1}{2i}\frac{1}{1 - (-\frac{z}{2i})} = \frac{i}{2}\sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^n z^n = \sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^{n+1} z^n.$$

Thus

$$\left[\sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^{n+1} z^n\right]' = \sum_{n=1}^{\infty} n\left(\frac{i}{2}\right)^{n+1} z^{n-1} = \sum_{m=0}^{\infty}(m + 1)\left(\frac{i}{2}\right)^{m+2} z^m.$$

Consequently the required expansion is

$$f(z)=\sum_{n=1}^{\infty}\frac{i^{n-1}}{z^n}+\sum_{n=0}^{\infty}(n+1)\left(\frac{i}{2}\right)^{n+2}z^n, \qquad 1<|z|<2.$$

Solution 4: The principal branch of the logarithm maps bijectively D onto the square S. Hence $f(z)$ has a simple pole at $\operatorname{Log} w$ and

$$\begin{aligned}\operatorname{Res}[f(z),\operatorname{Log} w] &= \lim_{z\to\operatorname{Log} w}(z-\operatorname{Log} w)f(z)=\lim_{z\to\operatorname{Log} w}\frac{z-\operatorname{Log} w}{e^z-w}ze^z\\ &= \lim_{z\to\operatorname{Log} w}\frac{1}{\frac{e^z-w}{z-\operatorname{Log} w}}ze^z=\frac{1}{e^{\operatorname{Log} w}}\cdot\operatorname{Log} w\cdot e^{\operatorname{Log} w}=\operatorname{Log} w.\end{aligned}$$

Now the statement follows from the residue theorem.

Solution 5: First note that $\cos 2x=\cos^2 x-\sin^2 x$ which validates the hint. Let

$$f(z)=\frac{1-e^{2iz}}{z^2}.$$

This is an even function if z is real. The numerator has a simple zero at 0, and hence f has a simple pole at 0 with

$$\operatorname{Res}[f,0]=\lim_{z\to 0}zf(z)=\left[-e^{2iz}\right]'_{z=0}=-2i.$$

Let $0<r<R$. Let $\gamma_\varrho(\theta)=\varrho e^{i\theta}$, where $\theta\in[0,\pi]$ and $\varrho>0$ is a constant, be the standard parametrization of upper semicircle with center at 0 and radius ϱ. By the fractional residue theorem

$$\lim_{t\to 0}\int_{\gamma_r}f(z)dz=\pi i\operatorname{Res}[f,0]=2\pi.$$

Also

$$\left|\int_{\gamma_R}f(z)dz\right|\le\int_{\gamma_R}|f(z)||dz|\le\frac{1}{R^2}\pi R+\frac{1}{R^2}\underbrace{\int_0^\pi e^{-2R\sin\theta}d\theta}_{<\frac{\pi}{2R}}\longrightarrow 0 \text{ as } R\to\infty.$$

Consequently, if the integral we seek is equal denoted by I, then

$$\operatorname{Re}\left[\lim_{r\to 0,R\to\infty}\int_{[-R,-r]\cup(-\gamma_r)\cup[r,R]\cup\gamma_R}f(z)dz\right]=2I-2\pi.$$

Solution 6: The polynomial z^4 has a zero of order 4 at the origin. On the circle $\partial D(0,3)$ we have

$$|z^4| = 81 > 70 = 54 + 16 \geq |-2iz^3 + 16|.$$

By Rouche's theorem the polynomials z^4 and $p(z) = z^4 - 2iz^3 + 16$ have the same number of roots in the disc $D(0,3)$. Let $R > 0$. If $-r$ changes within the interval $[-R, 0]$ (from left to right), then $p(-r)$ stays in the first quadrant, and the change of argument of $p(-r)$ is very small if R is large enough. Indeed, if α_r denotes the angle between the nonnegative semiaxis and the line segment $[0, p(-r)]$, then

$$\tan \alpha_r = \frac{2r^3}{r^4 + 16} \longrightarrow 0 \text{ as } r \to \infty.$$

If $r \in [0, R]$, then $p(-ir) > 0$, so there is no increase in argument. If $z = Re^{i\theta}$ changes as θ moves in $[-\pi, -\pi/2,]$ from right to left, then z^4 makes full revolution around the circle $\partial D(0, R^4)$, whereas the remaining terms of $p(z)$ create only a very small disturbance if R is large enough. This is so, because

$$\lim_{R\to\infty} \frac{|-2iR^3 e^{3i\theta} + 16|}{R^4} = 0.$$

The argument principle implies that there is only one zero in the third quadrant.

Solution 7: The first two properties imply that

$$f(z) = \frac{(z^2+1)^2}{(z-1)^3} h(z),$$

with some analytic function $h : \mathbb{C} \longrightarrow \mathbb{C}$ such that $h(\pm i) \neq 0$ and $h(1) \neq 0$. The third property implies that h has a finite limit $c \neq 0$ at ∞, and thus h is constant (by Liouville's theorem) and

$$f(z) = c\frac{(z^2+1)^2}{(z-1)^3}.$$

Since

$$\text{Res}\,[f, 1] = \frac{1}{2}\left[c(z^2+1)^2\right]''_{z=1} = 4c,$$

it follows that

$$f(z) = \frac{(z^2+1)^2}{2(z-1)^3}.$$

The given conditions determine the function f uniquely.

Solution 8: By Cauchy's estimates if $R > 0$ and $n \geq 1$, then

$$\frac{|f^{(n)}(0)|}{n!} \leq \frac{A + \sqrt{R}}{R^n} \longrightarrow 0$$

as $R \to \infty$. Hence all derivatives of f at 0 are zero, so the Taylor's series expansion of f reduces to the constant term.